Physics 401 **Fall 2003**

Homework Set #10 – Solutions

9.23 A particle of mass m_1 and velocity u_1 collides with a particle of mass m_2 at rest. The two particles stick together. What fraction of the original kinetic energy is lost in the collision?

Since the two particles stick together, conservation of momentum yields

$$m_1\vec{u}_1 = (m_1 + m_2)\vec{v}$$

where $\vec{v}$ is the final velocity of the stuck particles. This indicates that $\vec{u}_1$ and $\vec{v}$ are proportional to each other, so that the collision takes place along a line (one-dimensional). Squaring this equation yields

$$m_1^2 u_1^2 = (m_1 + m_2)^2 v^2 \qquad \Rightarrow \qquad \left(\frac{v}{u_1}\right)^2 = \left(\frac{m_1}{m_1 + m_2}\right)^2$$

The fraction of kinetic energy lost is

$$\frac{\Delta T}{T} = \frac{T_0 - T_f}{T_0} = 1 - \frac{T_f}{T_0} = 1 - \frac{\frac{1}{2}(m_1 + m_2)v^2}{\frac{1}{2}m_1 u_1^2} = 1 - \left(\frac{m_1 + m_2}{m_1}\right)^2 \left(\frac{v}{u_1}\right)^2$$
$$= 1 - \frac{m_1}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}$$

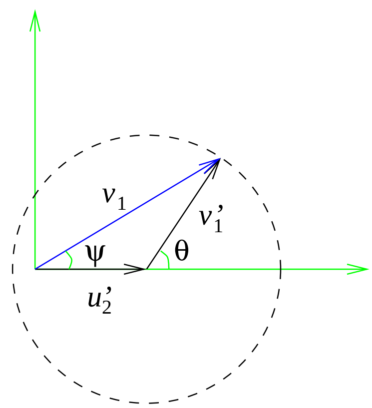
Note that in the limit $m_1 \to 0$ (very light projectile), 100% of the kinetic energy is lost, since the particles stick together and basically do not move. On the other hand, for $m_1 \to \infty$, the projectile is so heavy it just plows into the target and keeps on going. In this case, the energy loss approaches 0%.

9.35 A particle of mass m_1 with initial laboratory velocity u_1 collides with a particle of mass m_2 at rest in the LAB system. The particle m_1 is scattered through a LAB angle ψ and has a final velocity v_1, where $v_1 = v_1(\psi)$. Find the surface such that the time of travel of the scattered particle from the point of collision to the surface is independent of the scattering angle. Consider the cases

a) $m_2 = m_1$,

Let us first consider this for general masses. What this surface is may be thought of as the 'curve' (or surface when looked at in three dimensions) $r(\psi)$ such that the time it takes to go from $r = 0$ (the point of collision) to $r(\psi)$ takes a fixed amount of time, say t. Since $r(\psi) = v_1(\psi)t$, this means we really want to find the velocity $v_1(\psi)$. Finding this velocity as a function of the LAB angle ψ is the goal of this problem. Note that here we are using the textbook definitions of the scattering quantities.

In this case, the final LAB velocity of m_1 is the vector sum of its CM velocity, $\vec{v}_1'$ and the center of mass velocity, $\vec{u}_2'$ with respect to the LAB.



Recall that $u'_2 = (m_1/m_2)u'_1$ *and* $v'_1 = u'_1$ *(for an elastic collision) where* $u'_1 = u_1/(1 + m_1/m_2)$. *Since we are not changing the initial velocity* u_1, *this indicates that both* u'_2 *and* v'_1 *in the figure have fixed magnitudes; the only quantity that may change is the CM scattering angle* θ. *As* θ *varies, it is clear that the LAB velocity* v_1 *traces out the circumference of a circle . As a result, the constant* v_1 *surface is simply a circle (sphere in three dimensions), however with its center offset from the origin. Since the distance traveled is directly proportional to the velocity, this indicates that the fixed time surface is a circle (or sphere) with center offset from the point of collision.*

In equations, we may write down the vector components (in two dimensions)

$$v_{1\,x} = u'_2 + v'_1 \cos\theta = u'_1 \left(\cos\theta + \frac{m_1}{m_2}\right)$$

$$v_{1\,y} = v'_1 \sin\theta = u'_1 \sin\theta$$

Taking the ratio $v_{1\,y}/v_{1\,x}$ *gives the familiar relation* $\tan\psi = \sin\theta/(\cos\theta + m_1/m_2)$. *However, we are mainly interested in the LAB velocity* $\vec{v}_1$ *and do not really care about the CM scattering angle* θ. *As a result, we may eliminate* θ *using the trig relation* $\sin^2\theta + \cos^2\theta = 1$. *In particular*

$$\sin\theta = \frac{v_{1\,y}}{u'_1}, \qquad \cos\theta = \frac{v_{1\,x}}{u'_1} - \frac{m_1}{m_2} \qquad \Rightarrow \qquad \left(\frac{v_{1\,x}}{u'_1} - \frac{m_1}{m_2}\right)^2 + \left(\frac{v_{1\,y}}{u'_1}\right)^2 = 1$$

Multiplying by t^2 *(where* t *is the fixed time that we wait) and using* $u'_1 = u_1/(1 + m_1/m_2)$, $x = v_{1\,x}t$, *etc., we see that this equation describes a circle*

$$\left(x - \frac{m_1}{m_1 + m_2} u_1 t\right)^2 + y^2 = \left(\frac{m_2}{m_1 + m_2} u_1 t\right)^2 \tag{1}$$

For $m_2 = m_1$, *this equation reduces to*

$$(x - \tfrac{1}{2} u_1 t)^2 + y^2 = (\tfrac{1}{2} u_1 t)^2$$

which is that of a circle passing through the origin. In polar coordinates, this is

$$r(\psi) = (u_1 t) \cos\psi$$

b) $m_2 = 2m_1$, and

Working in rectangular coordinates, and using (1)*, the surface is given by*

$$(x - \tfrac{1}{3}u_1 t)^2 + y^2 = (\tfrac{2}{3}u_1 t)$$

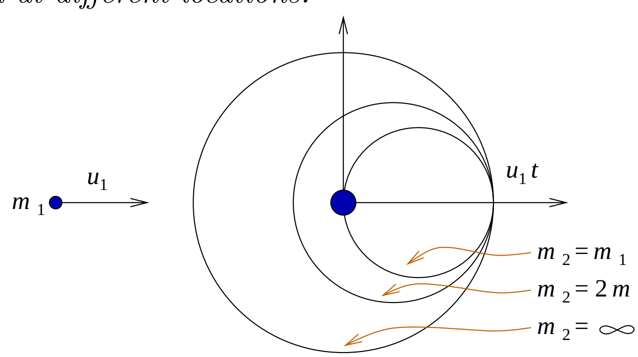
This time, the displacement of the center ($\frac{1}{3}u_1 t$ to the right) is less than the radius ($\frac{2}{3}u_1 t$). Hence the constant time surface is still a circle, however this time no longer passing through the origin. Note that this expression may be converted to polar coordinates. However, the result is messy and rather uninspiring.

c) $m_2 = \infty$.

For $m_2 = \infty$, the circle equation (1) *simplifies considerably. The result is $x^2 + y^2 = (u_1 t)^2$, which yields a circle of radius $u_1 t$ centered at the origin. This of course has the trivial polar coordinates expression $r(\psi) = u_1 t$. This case is of course easy to understand. If the target is infinitely heavy, it cannot move. Then conservation of energy indicates that $v_1 = u_1$ independent of the scattering angle. For particles leaving the origin at constant velocity, the equal time surfaces must be circles. (Momentum conservation still holds in this case, courtesy of an infinitely massive m_2, that can 'soak up' any finite amount of momentum without actually moving.)*

Suggest an application of this result in terms of a detector for nuclear particles.

As we have seen, in all three parts, the constant time surfaces are circles (spheres), just centered at different locations.

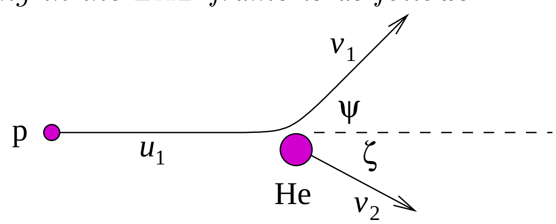


As a possible application, let us suppose we are shooting a known beam of particles (perhaps neutrons) against a known target. This means we know the masses m_1 and m_2. However, perhaps we do not know the velocity u_1, so we would like to design an apparatus that can measure this velocity. There are various ways to measure velocities. But a straightforward method is to measure the time of flight for a particle to go a certain distance.

If we do not care about the scattered angle, but only care about measuring the velocity u_1, then we could build a detector with the shape of a sphere (or a partial sphere) to capture as many particles as possible. Since the sphere is a constant time surface (assuming we place the center in the correct place, depending on the masses), this means we may use the time of flight to the detector without incorporating any angular correction factors.

9.43 A proton (mass m) of kinetic energy T_0 collides with a helium nucleus (mass $4m$) at rest. Find the recoil angle of the helium if $\psi = 45°$ and the inelastic collision has $Q = -T_0/6$.

The scattering in the LAB frame is as follows



As a result, conservation of momentum gives

$$mu_1 = \tfrac{1}{\sqrt{2}}mv_1 + (4m)v_2\cos\zeta$$
$$0 = \tfrac{1}{\sqrt{2}}mv_1 - (4m)v_2\sin\zeta$$

where we have used the fact that $\psi = 45°$. Since this is an inelastic collision, energy is not conserved. On the other hand, since we know how much energy is lost, we may write

$$\tfrac{1}{2}mu_1^2 - \tfrac{1}{6}T_0 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}(4m)v_2^2$$

Using the fact that $T_0 = \frac{1}{2}mu_1^2$, and canceling an overall factor of m, the above three equations may be rewritten as

$$u_1 = \tfrac{1}{\sqrt{2}}v_1 + 4v_2\cos\zeta, \qquad \tfrac{1}{\sqrt{2}}v_1 = 4v_2\sin\zeta$$
$$\tfrac{5}{6}u_1^2 = v_1^2 + 4v_2^2 \tag{2}$$

Since the problem specifies T_0, we may assume u_1 is known. Then all we have to do is to solve these three equations for the recoil angle ζ.

Let us actually start by eliminating ζ from the first two equations of (2)*. While in the end, we actually want ζ, this first step allows us to find the magnitudes of the scattered velocities*

$$\left\{\begin{array}{l} 4v_2\cos\zeta = u_1 - \frac{1}{\sqrt{2}}v_1 \\ 4v_2\sin\zeta = \frac{1}{\sqrt{2}}v_1 \end{array}\right\} \qquad \Rightarrow \qquad 16v_2^2 = u_1^2 + v_1^2 - \sqrt{2}u_1v_1$$

Combining this with the last equation of (2) *now gives*

$$\left\{\begin{array}{l} \frac{5}{6}u_1^2 = v_1^2 + 4v_2^2 \\ 16v_2^2 = u_1^2 + v_1^2 - \sqrt{2}u_1v_1 \end{array}\right\} \qquad \Rightarrow \qquad 5v_1^2 - \sqrt{2}v_1u_1 - \tfrac{7}{3}u_1^2 = 0$$

$$\Rightarrow \qquad v_1 = \left(\frac{3\sqrt{2} + \sqrt{438}}{30}\right)u_1$$

Note that we have picked the positive root of the solution to the quadratic equation since we know that v_1 is positive. Inserting this back into either equation yields

$$v_2^2 = \frac{49 - 2\sqrt{219}}{600} u_1^2$$

Finally, we may return to the angle ζ. Looking at (2), we may compute

$$\sin\zeta = \frac{v_1}{4\sqrt{2}v_2} = \frac{3\sqrt{2} + \sqrt{438}}{4\sqrt{3}(40 - 2\sqrt{219})^{1/2}} \approx 0.825 \qquad \Rightarrow \qquad \zeta \approx 55.6^\circ$$

While this result is somewhat ugly as far as the square roots are concerned, it is interesting to see that the angle is independent of the initial kinetic energy of the proton. This is only true because the energy loss Q is given as a fraction of the initial kinetic energy T_0.

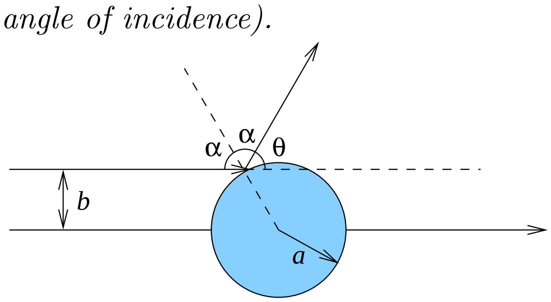
9.46 Calculate the differential cross section $\sigma(\theta)$ and the total cross section σ_t for the elastic scattering of a particle from an impenetrable sphere; the potential is given by

$$U(r) = \begin{cases} 0, & r > a \\ \infty, & r < a \end{cases}$$

Recall that the differential cross section may be computed in terms of the impact parameter and scattering angle as

$$\sigma(\theta) = \frac{b(\theta)}{\sin\theta} \left| \frac{db(\theta)}{d\theta} \right| \tag{3}$$

Thus our goal is to find $b(\theta)$ for this problem. For hard spheres, it is best to ignore the explicit potential $U(r)$, which goes to infinity inside the sphere (indicating impenetrability). Instead, we can just perform specular reflection (angle of reflection = angle of incidence).



Using a bit of trig, we see that the impact parameter b may be related to the incident angle α through $\sin\alpha = b/a$. Also, the scattering angle θ is given by $\theta = \pi - 2\alpha$. Hence

$$b = a\sin\alpha = a\sin(\tfrac{\pi}{2} - \tfrac{\theta}{2}) = a\cos(\tfrac{\theta}{2})$$

Inserting this into the cross section equation yields

$$\sigma(\theta) = \frac{a\cos(\frac{\theta}{2})}{\sin\theta}\left|-\frac{a}{2}\sin(\tfrac{\theta}{2})\right| = \tfrac{1}{4}a^2$$

Note that this differential cross section is independent of the scattering angle. This indicates that the hard sphere scattering is isotropic. Integrating over the solid angle yields a total cross section

$$\sigma_t = \pi a^2 \tag{4}$$

which is the same as the cross sectional area of the sphere. This is known as the geometrical cross section.

Finally, this problem is interesting to revisit from a quantum mechanical viewpoint, where low energy scattering yields $\sigma_t = 4\pi a^2$, *which is four times as large! One may expect to obtain the classical result* (4) *when looking at high energy scattering. However, even in this 'classical limit', the quantum result is* $\sigma_t = 2\pi a^2$ *and is twice as large. In other words, the quantum mechanical result is always larger than the classical one. This peculiarity arises because of the infinitely sharp potential barrier, which is not necessarily realistic, and which causes funny effects in wave mechanics.*