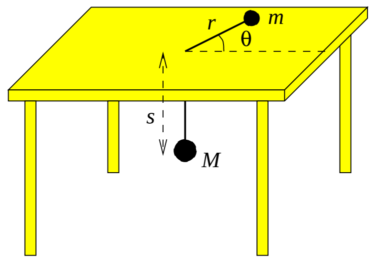
Physics 401 **Fall 2003**

Midterm II – Solutions

[35 pts] 1. A mass m lies on a table (without friction), connected to another mass M through an ideal string running through a hole in the center of the table. The total length of the string, $r+s$, is constant.



[10] *a*) Write down the Lagrangian for the system in terms of r, θ and s and their time derivatives. Assume that r and s are independent, but introduce an auxiliary condition $f(r,s)=0$ to enforce the constraint that the total length of the string is constant.

Let us start by assuming as the problem says that r and s are independent. In this case, the masses m and M move independently. The former is free to move horizontally on the table (in both the r and θ direction), while the latter moves up and down. The motion of mass m is described by the free particle Lagrangian in polar coordinates

$$L_m = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

On the other hand, the mass M moves under the influence of gravity. We assume it does not swing from side to side. Since the s coordinate increases as the mass is lowered, the potential energy is $U=-Mgs$, and the Lagrangian is given by

$$L_M = \tfrac{1}{2}M\dot{s}^2 + Mgs$$

The Lagrangian for the system is then

$$L = \tfrac{1}{2}M\dot{s}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mgs \tag{1}$$

However, we must also incorporate a constraint that $r+s=a$. Here we have let a denote the total length of the string. This may be done by defining the auxiliary condition

$$f(r,s) = r+s-a \tag{2}$$

Hence the system is given by the Lagrangian (1) and the auxiliary condition (2).

We can, of course, explicitly incorporate the auxiliary condition into the Lagrangian itself. The result is

$$L = \tfrac{1}{2}M\dot{s}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mgs + \lambda(r+s-a) \tag{3}$$

where λ is the Lagrange multiplier.

[15] b) By making use of the constraint (and eliminating the Lagrange multiplier), write down the equations of motion in terms of the r and θ coordinates only. Show that the angular momentum for mass m about the center of the table is conserved.

Let us work with the Lagrangian (3). *Note that, for the Lagrange multiplier method to work, we must treat r and s as independent variables in the Lagrangian. We only worry about the constraint after deriving the equations of motion. For the Lagrangian* (3)*, the r and s equations of motion are*

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + \lambda, \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \Rightarrow \qquad m\ddot{r} = mr\dot{\theta}^2 + \lambda \tag{4}$$

$$\frac{\partial L}{\partial s} = Mg + \lambda, \qquad \frac{\partial L}{\partial \dot{s}} = M\dot{s} \qquad \Rightarrow \qquad M\ddot{s} = Mg + \lambda \tag{5}$$

while for the θ equation of motion, we note that θ is cyclic. Therefore

$$\frac{\partial L}{\partial \dot{\theta}} = \textit{constant} \qquad \Rightarrow \qquad mr^2\dot{\theta} = \ell \tag{6}$$

where ℓ denotes the constant angular momentum. This fact demonstrates that angular momentum is conserved for this problem.

We now eliminate λ from the equations of motion. Subtracting (5) *from* (4) *gives $m\ddot{r} - M\ddot{s} = mr\dot{\theta}^2 - Mg$. However, now we may use the constraint $r + s = a$. Taking two time derivatives gives $\ddot{s} = -\ddot{r}$. As a result, we find the equation of motion*

$$(M + m)\ddot{r} = mr\dot{\theta}^2 - Mg$$

If desired, we may further remove $\dot{\theta}$ from this equation using conservation of angular momentum, (6). *The result is*

$$(M + m)\ddot{r} = \frac{\ell^2}{mr^3} - Mg \tag{7}$$

[10] c) Find the tension on the string (force of constraint) in terms of r and ℓ, where ℓ is the conserved angular momentum of mass m.

Note that the tension on the string is given by the force of constraint equation

$$T = -Q_r = -\lambda\frac{\partial f}{\partial r} = -\lambda$$

Here Q_r denotes the force of the string on mass m in the radial direction. This force is towards the center of the table, or in the minus r direction. However, we take the tension T to be positive. This is the reason for the minus sign above.

In any case, to find the tension, we need to solve for the Lagrange multiplier. From (4)*, we have*

$$T = -\lambda = mr\dot{\theta}^2 - m\ddot{r} = \frac{\ell^2}{mr^3} - m\ddot{r}$$

Substituting in (7) for $\ddot{r}$ then yields

$$T = \frac{\ell^2}{mr^3} - \frac{m}{M+m}\left(\frac{\ell^2}{mr^3} - Mg\right) = \frac{M}{M+m}\left(mg + \frac{\ell^2}{mr^3}\right)$$

[35 pts] 2. Suppose two bodies undergo central force motion with a force $F = -k/r^3$ ('inverse cube force law').

[10] *a*) Show that if $\ell^2 < \mu k$ (where ℓ is the angular momentum) the two bodies will always fall into each other, regardless of the total energy of the system.

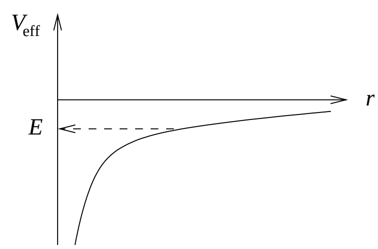
Let us examine the effective potential for this problem. The potential energy may be obtained by integrating the force

$$U(r) = -\int F(r)\,dr = \int \frac{k}{r^3}dr = -\frac{k}{2r^2}$$

As a result

$$V_{\text{eff}}(r) = U(r) + \frac{\ell^2}{2\mu r^2} = \left(\frac{\ell^2}{\mu} - k\right)\frac{1}{2r^2}$$

Note that for this inverse cube force, both the actual potential energy and the centrifugal potential behave as $1/r^2$. As a result, the nature of the effective potential depends on which term dominates. For $\ell^2 < \mu k$, we see that V_{eff} is negative, and looks more or less like



It ought to be clear that, no matter what energy E, the two bodies will always reach $r = 0$, and hence fall into each other.

[10] *b*) Show that $r = 1/(a + b\theta)$ is a solution to this problem when $\ell^2 = \mu k$ (a and b are constants).

Let us look at the equation for $r(\theta)$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r) \qquad \Rightarrow \qquad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{\mu k}{\ell^2}\frac{1}{r}$$

This equation is quite simple when written in terms of $u = 1/r$:

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu k}{\ell^2}\right)u = 0 \tag{8}$$

For $\ell^2 = \mu k$, this equation is almost trivial

$$\frac{d^2u}{d\theta^2} = 0$$

This is clearly satisfied for

$$u = \frac{1}{r} = a + b\theta$$

[15] c) For $\ell^2 > \mu k$, the bodies will not collide. Solve for the orbital path $r(\theta)$ for this case. Make any convenient choice of initial conditions, but specify the physical limits on θ.

It is perhaps easiest to work with the equation (8) that we derived above. For $\ell^2 > \mu k$, we may define

$$\omega_0 = \sqrt{1 - \frac{\mu k}{\ell^2}}$$

so that we must solve

$$\frac{d^2u}{d\theta^2} + \omega_0^2 u = 0$$

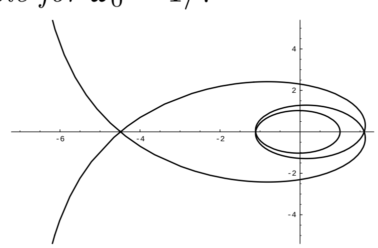
The solution to this equation is just like that of a harmonic oscillator (even though this is not a harmonic oscillator problem). Since we are free to choose initial conditions, let us write

$$u = A\cos\omega_0\theta \qquad \Rightarrow \qquad r = \frac{1}{A\cos\omega_0\theta}$$

Note that r takes on its minimum value $1/A$ when $\theta = 0$, while r goes to infinity when $\cos\omega_0\theta = 0$. This occurs at $\theta = \pm\pi/(2\omega_0)$. Hence the physical range of θ should be limited to

$$-\frac{\pi}{2\omega_0} < \theta < \frac{\pi}{2\omega_0}$$

Since ω_0 can be a very small number, this indicates that θ can actually go around the circle (2π) many times. As a result, typical orbits are spirals, known as Cotes' spirals. Here is an example for $\omega_0 = 1/7$

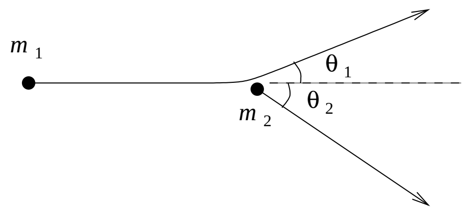


Note that this part may also be solved by integrating the equation

$$\theta(r) = \pm\int \frac{\ell/r^2\,dr}{\sqrt{2\mu(E - V_{\text{eff}}(r))}}$$

This gives a more direct relation between the constant A and the energy E of the orbit. However, the integral is slightly more tedious (it is best performed by changing variables $u = 1/r$).

[30 pts] 3. A particle of mass m_1 collides elastically with an initially stationary particle of mass m_2. The respective particles emerge after the collision with angles θ_1 and θ_2 in the LAB frame.



[10] *a*) Show that θ_2 can never exceed $90°$.

First of all, we ought to note that physically it would be very bizarre if $\theta_2 > 90°$. This case would correspond to shooting m_1 at m_2 and then finding that m_2 actually flies backward towards us. Something ought to be wrong with this picture....

Let us use the same notation as Marion (except that we use θ_1 and θ_2 as indicated). Then conservation of momentum in the x direction yields

$$m_1 u_1 = m_1 v_{1\,x} + m_2 v_{2\,x} \qquad \text{or} \qquad m_1 u_1 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \tag{9}$$

If $\theta_2 > 90°$ then $\cos\theta_2 < 0$, so the last term in (9) would be negative. This would imply that $m_1 u_1 < m_1 v_1 \cos\theta_1$ or $u_1 < v_1 \cos\theta_1$. However, $\cos\theta_1$ cannot exceed 1. Hence we can conclude that $u_1 < v_1$ whenever $\theta_2 > 90°$. Turning next to the conservation of energy, we have

$$m_1 u_1^2 = m_1 v_1^2 + m_2 v_2^2 \qquad \Rightarrow \qquad m_1 u_1^2 \ge m_1 v_1^2 \qquad \Rightarrow \qquad u_1 \ge v_1$$

which is the simple statement that the final velocity of m_1 cannot exceed the initial velocity of m_1 for an elastic collision. This is clearly in contradiction with $u_1 < v_1$ when $\theta_2 > 90°$. Hence we may conclude that θ_2 cannot exceed $90°$.

Another way to prove this is to use the relation between CM and LAB angles

$$\tan\theta_1 = \frac{\sin\theta_{\mathrm{CM}}}{\cos\theta_{\mathrm{CM}} + (m_1/m_2)}, \qquad \tan\theta_2 = \frac{\sin\theta_{\mathrm{CM}}}{1 - \cos\theta_{\mathrm{CM}}} \tag{10}$$

The latter equation is equivalent to the statement

$$\tan\theta_2 = \cot\frac{\theta_{\mathrm{CM}}}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta_{\mathrm{CM}}}{2}\right) \qquad \Rightarrow \qquad \theta_2 = \frac{\pi - \theta_{\mathrm{CM}}}{2} \tag{11}$$

Since θ_{CM} must lie between 0 and π, this shows that θ_2 must be between 0 and $90°$ degrees.

[20] *b*) Find the ratio m_1/m_2 in terms of θ_1 and θ_2.

Let us use the relations (10) *and* (11)*. Solving the latter for* $\theta_{\rm CM}$ *yields* $\theta_{\rm CM} = \pi - 2\theta_2$*. Substituting this into the first equation of* (10) *then gives*

$$\tan\theta_1 = \frac{\sin(\pi - 2\theta_2)}{\cos(\pi - 2\theta_2) + (m_1/m_2)} = \frac{\sin 2\theta_2}{-\cos 2\theta_2 + (m_1/m_2)}$$

This may easily be solved for the ratio m_1/m_2

$$\frac{m_1}{m_2} = \cos 2\theta_2 + \sin 2\theta_2 \cot\theta_1$$