Physics 512 **Winter 2003**

Homework Set #2 – Due Friday, January 24

1. We may add three angular momenta, $\vec{J}_1$, $\vec{J}_2$ and $\vec{J}_3$, by first adding the first two, $\vec{J}_{12} = \vec{J}_1 + \vec{J}_2$, and then by adding this result to the last one, $\vec{J}_{123} = \vec{J}_{12} + \vec{J}_3$. Using this (or any other suitable method), add three spin-1/2 states and write down the resulting coupled $|jm\rangle$ states in terms of the uncoupled basis. Give a complete set of commuting operators that are diagonal in the coupled basis. [Note that $|\frac{3}{2}\,\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$ should be an obvious state.]

2. From group theory, the character χ of a representation $\mathcal{D}$ is defined by $\chi = \operatorname{Tr}\mathcal{D}$. In particular, for a spin-j representation of the rotation group

$$\chi^{(j)}(\phi) = \sum_{m=-j}^{j} \langle jm|R(\phi\hat{n})|jm\rangle$$

where $R(\phi\hat{n})$ is the rotation operator corresponding to a rotation by angle ϕ about the $\hat{n}$ axis.

   a) Prove that $\chi^{(j)}(\phi)$ is independent of the axis of rotation, $\hat{n}$.

   b) As a result, we may compute $\chi^{(j)}(\phi)$ by considering a rotation by ϕ about the z-axis, $R(\phi\hat{z})$. Show that

$$\chi^{(j)}(\phi) = \frac{\sin(j+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}$$

3. This is similar to Sakurai, Chapter 3, Problem 22. Consider a system with angular momentum $j = 1$.

   a) Prove that (only for $j = 1$ states) the operator

$$\mathcal{O} = J_y(J_y - \hbar)(J_y + \hbar)$$

   is equivalent to the null operator. As a result, show that it is legitimate to take $J_y^3 = \hbar^2 J_y$.

   b) Show that (for $j = 1$) we may write

$$\exp\left(-\frac{i}{\hbar}J_y\beta\right) = 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

   and, as a result, derive the representation matrix $D^{(j=1)}_{m'm}(\alpha,\beta,\gamma)$.

4. The relation between spherical and cartesian components of a vector $\vec{V}$ is given by

$$V_0^{(1)} = V_z, \qquad V_{\pm 1}^{(1)} = \mp \frac{1}{\sqrt{2}}(V_x \pm iV_y)$$

We may take a tensor product of two vectors $V_q^{(1)}$ and $W_q^{(1)}$ according to

$$T_q^{(k)} = \sum_{q_1, q_2} \langle 11; q_1 q_2 | 11; kq \rangle V_{q_1}^{(1)} W_{q_2}^{(1)}$$

[this is a special case of Sakurai (3.10.27)]

a) Show that $T_0^{(0)}$ is proportional to the scalar product $\vec{V} \cdot \vec{W}$.

b) Show that $T_q^{(1)}$ is proportional to the (spherical tensor) components of $(\vec{V} \times \vec{W})$.

c) Write down the five components of $T_q^{(2)}$ in terms of $V_q^{(1)}$ and $W_q^{(1)}$.

d) Find $T_0^{(2)}$ in terms of the cartesian components V_x, V_y, V_z and W_x, W_y, W_z.

[Note that the answer is given by Sakurai (3.10.26), which incidentally has a typo in the first equation. However this problem is basically to prove those equations.]