Physics 512 **Winter 2003**

Homework Set #2 – Solutions

1. We may add three angular momenta, $\vec{J}_1$, $\vec{J}_2$ and $\vec{J}_3$, by first adding the first two, $\vec{J}_{12} = \vec{J}_1 + \vec{J}_2$, and then by adding this result to the last one, $\vec{J}_{123} = \vec{J}_{12} + \vec{J}_3$. Using this (or any other suitable method), add three spin-1/2 states and write down the resulting coupled $|jm\rangle$ states in terms of the uncoupled basis. Give a complete set of commuting operators that are diagonal in the coupled basis. [Note that $|\frac{3}{2}\,\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$ should be an obvious state.]

   We start by adding the first two angular momenta as suggested:

$$
\begin{aligned}
|11\rangle &= |\uparrow\uparrow\rangle \\
|10\rangle &= \tfrac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big) \qquad\qquad |00\rangle = \tfrac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big) \\
|1\,{-1}\rangle &= |\downarrow\downarrow\rangle
\end{aligned}
$$

   (where it should be understood that these apply to the first two spins). Now, when we add the third spin to this, we have to make a choice. Either we can add it to the triplet or the singlet. Adding it to the singlet is trivial, and gives

$$
\begin{aligned}
|\tfrac{1}{2}\,\tfrac{1}{2}\rangle &= |00\rangle \otimes |\uparrow\rangle = \tfrac{1}{\sqrt{2}}\big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big) \\
|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle &= |00\rangle \otimes |\downarrow\rangle = \tfrac{1}{\sqrt{2}}\big(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle\big)
\end{aligned}
\tag{1}
$$

   We now turn to the triplet. When adding the third spin to the triplet, we can end up with either $j = \frac{1}{2}$ or $j = \frac{3}{2}$. Looking up the Clebsch-Gordan coefficients, we find

$$
\begin{aligned}
|\tfrac{3}{2}\,\tfrac{3}{2}\rangle &= |11\rangle \otimes |\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle \\
|\tfrac{3}{2}\,\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}\big(|11\rangle \otimes |\downarrow\rangle + \sqrt{2}|10\rangle \otimes |\uparrow\rangle\big) = \tfrac{1}{\sqrt{3}}\big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\big) \\
|\tfrac{3}{2}\,{-\tfrac{1}{2}}\rangle &= \tfrac{1}{\sqrt{3}}\big(\sqrt{2}|10\rangle \otimes |\downarrow\rangle + |1\,{-1}\rangle \otimes |\uparrow\rangle\big) = \tfrac{1}{\sqrt{3}}\big(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\big) \\
|\tfrac{3}{2}\,{-\tfrac{3}{2}}\rangle &= |1\,{-1}\rangle \otimes |\downarrow\rangle = |\downarrow\downarrow\downarrow\rangle
\end{aligned}
\tag{2}
$$

   for resulting angular momentum $\frac{3}{2}$, and

$$
\begin{aligned}
|\tfrac{1}{2}\,\tfrac{1}{2}\rangle' &= \tfrac{1}{\sqrt{3}}\big(\sqrt{2}|11\rangle \otimes |\downarrow\rangle - |10\rangle \otimes |\uparrow\rangle\big) = \tfrac{1}{\sqrt{6}}\big(2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big) \\
|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle' &= \tfrac{1}{\sqrt{3}}\big(|10\rangle \otimes |\downarrow\rangle - \sqrt{2}|1\,{-1}\rangle \otimes |\uparrow\rangle\big) = \tfrac{1}{\sqrt{6}}\big(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle - 2|\downarrow\downarrow\uparrow\rangle\big)
\end{aligned}
\tag{3}
$$

   for resulting angular momentum $\frac{1}{2}$. Note that the prime is to indicate that this set of $j = \frac{1}{2}$ states is different from the first one found above.

   This type of degeneracy is quite common whenever we add three or more angular momenta together. One way to express this is that when we combine the first two spins, we have

$$\mathbf{\tfrac{1}{2}} \otimes \mathbf{\tfrac{1}{2}} = \mathbf{0} \oplus \mathbf{1}$$

Next we combine either $\mathbf{0}\otimes\mathbf{\frac{1}{2}}=\mathbf{\frac{1}{2}}$ *or* $\mathbf{1}\otimes\mathbf{\frac{1}{2}}=\mathbf{\frac{1}{2}}\oplus\mathbf{\frac{3}{2}}$*. So the final result is*

$$\mathbf{\tfrac{1}{2}}\otimes\mathbf{\tfrac{1}{2}}\otimes\mathbf{\tfrac{1}{2}}=\mathbf{\tfrac{1}{2}}\oplus\mathbf{\tfrac{1}{2}}\oplus\mathbf{\tfrac{3}{2}}$$

This decomposition also shows that you start with eight states (of spins up or down in the uncoupled basis), and end up with eight states as well (in the coupled basis), which is a good thing. While the two $\mathbf{\frac{1}{2}}$ *states are linearly independent, the explicit answers* (1) *and* (3) *are not necessarily unique. One could take appropriately normalized linear combinations of them (with the same* m *values, of course). This is just a symptom of a degeneracy in the states. Of course, the* $\mathbf{\frac{3}{2}}$ *states,* (2)*, are unique.*

This degeneracy between the two $\mathbf{\frac{1}{2}}$ *states indicates that by simplying saying "total spin-*1/2*" we have not fully accounted for a complete set of quantum numbers in the coupled basis. Of course, looking back at how we arrived at the coupled basis, we see that the first spin-*1/2 *state that we derived,* (1)*, came from the singlet of* $\vec{J}_{12}$*, while the second spin-*1/2 *state,* (3)*, came from the triplet of* $\vec{J}_{12}$*. This suggests that we may look at the eigenvalue of* J_{12}^2 *to distinguish among the coupled states. To be more precise, consider how we label the states by their quantum numbers. In the completely uncoupled basis, we start with a state described by six quantum numbers*

$$|j_1,j_2,j_3;m_1,m_2,m_3\rangle=|j_1m_1\rangle\otimes|j_2m_2\rangle\otimes|j_3m_3\rangle$$

Then after adding the first two angular momenta, we end up with $|j_1,j_2;j_{12},m_{12}\rangle\otimes|j_3;m_3\rangle$*. Finally, adding* $|j_{12},m_{12}\rangle$ *to* $|j_3,m_3\rangle$*, we end up with* $|j_1,j_2,j_3,j_{12};j,m\rangle$*. As a result, a complete set of commuting operators appropriate for our coupled basis are*

$$J_1^2,\quad J_2^2,\quad J_3^2,\quad J_{12}^2,\quad J_{123}^2,\quad J_{123\,z}$$

We start with six quantum numbers, and end with six, so everything fits together. Note that, in this scheme, J_{12}^2 *is singled out as one of the complete set of operators. Of course, this is not a unique choice, but happens to correspond to our choice of first adding* $\vec{J}_1$ *and* $\vec{J}_2$ *together. As a result, it provides the proper bookkeeping for the eight resulting states* (1)*,* (2) *and* (3)*.*

Another approach to this problem would be to start with the obvious state $|\frac{3}{2}\frac{3}{2}\rangle=|\uparrow\uparrow\uparrow\rangle$*, and then to repeatedly lower it by acting with* $J_{123\,-}$*. This provides the four states of the spin-*3/2 *combination of* (2)*. To get the remaining states, we use orthogonality. But here we run into the problem that there is no unique orthogonal* $|\frac{1}{2}\frac{1}{2}\rangle$ *state. In this case, we can make an arbitrary choice (or better yet choose it to be an eigenstate of* J_{12}^2*). This gives us one set of the spin-*1/2 *states. To get the second set, we simply demand orthogonality with respect to both the spin-*3/2 *states and the first spin-*1/2 *states.*

2. From group theory, the character χ of a representation $\mathcal{D}$ is defined by $\chi=\mathrm{Tr}\,\mathcal{D}$. In particular, for a spin-j representation of the rotation group

$$\chi^{(j)}(\phi)=\sum_{m=-j}^{j}\langle jm|R(\phi\hat{n})|jm\rangle$$

where $R(\phi\hat{n})$ is the rotation operator corresponding to a rotation by angle ϕ about the $\hat{n}$ axis.

a) Prove that $\chi^{(j)}(\phi)$ is independent of the axis of rotation, $\hat{n}$.

The above character expression is of course the trace of the $D^{(j)}$-matrix

$$\chi^{(j)}(\phi) = \sum_{m=-j}^{j} D^{(j)}_{mm}(\phi\hat{n}) = \operatorname{Tr} D^{(j)}(\phi\hat{n})$$

In order to show that $\chi^{(j)}(\phi)$ is independent of $\hat{n}$, we consider rotating the rotation axis $\hat{n}$ to a new rotation axis $\hat{n}'$ [so many 'rotations'!]. This is simply done by finding a rotation, say $R(\alpha\hat{m})$, that rotates $\hat{n}$ to $\hat{n}'$, i.e.

$$R(\phi\hat{n}') = R(\alpha\hat{m})R(\phi\hat{n})R^{\dagger}(\alpha\hat{m})$$

In general, finding α and $\hat{m}$ is a bit messy. However the good thing is that we will never need any explicit equations for α and $\hat{m}$. We just note that a choice of α and $\hat{m}$ always exists because of the group property of rotations. Taking a trace of both sides yields

$$\begin{aligned}
\chi^{(j)}_{\vec{n}'}(\phi) &= \operatorname{Tr} R(\phi\hat{n}') = \operatorname{Tr}[R(\alpha\hat{m})R(\phi\hat{n})R^{\dagger}(\alpha\hat{m})] \\
&= \operatorname{Tr}[R^{\dagger}(\alpha\hat{m})R(\alpha\hat{m})R(\phi\hat{n})] \qquad \textit{(cyclic property of the trace)} \\
&= \operatorname{Tr} R(\phi\hat{n}) = \chi^{(j)}_{\vec{n}}(\phi)
\end{aligned}$$

In this equation, we have explicitly labeled the character by the corresponding axis. However this shows that it is actually independent of the axis, thus completing the proof.

Incidentally, we prove the cyclic property of the trace of operators by inserting a complete set of states

$$\begin{aligned}
\operatorname{Tr}[AB] = \sum_m \langle jm|AB|jm\rangle &= \sum_{m,m'} \langle jm|A|jm'\rangle\langle jm'|B|jm\rangle \\
&= \sum_{m,m'} \langle jm'|B|jm\rangle\langle jm|A|jm'\rangle \\
&= \sum_{m'} \langle jm'|BA|jm'\rangle = \operatorname{Tr}[BA]
\end{aligned}$$

b) As a result, we may compute $\chi^{(j)}(\phi)$ by considering a rotation by ϕ about the z-axis, $R(\phi\hat{z})$. Show that

$$\chi^{(j)}(\phi) = \frac{\sin(j+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}$$

We have seen in class that a rotation about the $\hat{z}$-axis is straightforward

$$D^{(j)}_{m'm}(\phi\hat{z}) = \langle jm'|e^{-\frac{i}{\hbar}\phi J_z}|jm\rangle = e^{-im\phi}\delta_{mm'}$$

Taking the trace then gives

$$\chi^{(j)}(\phi) = \sum_{m=-j}^{j} D^{(j)}_{mm}(\phi\hat{z}) = \sum_{m=-j}^{j} e^{-im\phi}$$

Writing $z = e^{i\phi}$, we see that this is simply a geometric series

$$\phi^{(j)}(\phi) = z^{-j}\sum_{n=0}^{2j} z^n = z^{-j}\frac{z^{2j+1}-1}{z-1} = \frac{z^{j+\frac{1}{2}} - z^{-(j+\frac{1}{2})}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} = \frac{\sin(j+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}$$

3. This is similar to Sakurai, Chapter 3, Problem 22. Consider a system with angular momentum $j = 1$.

   a) Prove that (only for $j = 1$ states) the operator

$$\mathcal{O} = J_y(J_y - \hbar)(J_y + \hbar)$$

   is equivalent to the null operator. As a result, show that it is legitimate to take $J_y^3 = \hbar^2 J_y$.

   Any angular momentum $j = 1$ state may be decomposed in a $\vec{J}^2$, J_z basis:

$$|\chi\rangle = c_1|11\rangle + c_0|10\rangle + c_{-1}|1\,{-1}\rangle$$

   In this case, it is easy to see that

$$J_z(J_z - \hbar)(J_z + \hbar)|\chi\rangle = 0$$

   since the respective terms in the decomposition have J_z eigenvalues $-\hbar$, 0 and $\hbar$ respectively. To get from this J_z expression to the desired J_y expression, we can simply perform a rotation by $-\pi/2$ about the $\hat{x}$ axis

$$RJ_z(J_z - \hbar)(J_z + \hbar)R^\dagger R|\chi\rangle = 0$$

   or equivalently $\mathcal{O}|\chi'\rangle = 0$ where

$$\mathcal{O} = J_y(J_y - \hbar)(J_y + \hbar)$$

   and $|\chi'\rangle$ is a rotated state. Since $|\chi\rangle$ was completely arbitrary, so is $|\chi'\rangle$. Thus $\mathcal{O}$ vanishes when acting on arbitrary states (in the $j = 1$ subspace). Since $\mathcal{O}$

annihilates all ($j = 1$) states, it is equivalent to the null operator. Finally, setting $\mathcal{O} = 0$, we then multiply terms out to obtain $J_y^3 = \hbar^2 J_y$.

Another way to look at this is that we could have started more directly with a $\vec{J}^2$, J_y basis instead. Using $\hat{y}$ as the quantization axis, the proof follows immediately.

b) Show that (for $j = 1$) we may write

$$\exp\left(-\frac{i}{\hbar}J_y\beta\right) = 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

and, as a result, derive the representation matrix $D^{(j=1)}_{m'm}(\alpha,\beta,\gamma)$.

Using the result of a), expressed as $(J_y/\hbar)^3 = (J_y/\hbar)$, we write out the Taylor series expansion to obtain

$$\begin{aligned}
\exp\left(-\frac{i}{\hbar}J_y\beta\right) &= 1 - i\beta\left(\frac{J_y}{\hbar}\right) - \frac{\beta^2}{2!}\left(\frac{J_y}{\hbar}\right)^2 + i\frac{\beta^3}{3!}\left(\frac{J_y}{\hbar}\right)^3 + \frac{\beta^4}{4!}\left(\frac{J_y}{\hbar}\right)^4 - \cdots \\
&= 1 - i\beta\left(\frac{J_y}{\hbar}\right) - \frac{\beta^2}{2!}\left(\frac{J_y}{\hbar}\right)^2 + i\frac{\beta^3}{3!}\left(\frac{J_y}{\hbar}\right) + \frac{\beta^4}{4!}\left(\frac{J_y}{\hbar}\right)^2 - \cdots \\
&= 1 - i\left(\frac{J_y}{\hbar}\right)\left(\beta - \frac{\beta^3}{3!} + \cdots\right) + \left(\frac{J_y}{\hbar}\right)^2\left(-\frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots\right) \\
&= 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta + \left(\frac{J_y}{\hbar}\right)^2(\cos\beta - 1)
\end{aligned}$$

where we have identified the Taylor series expansions for $\sin\beta$ *and* $\cos\beta$.

We now go on to calculate the $d^{(1)}_{m'm}(\beta)$ function using the matrix representation

$$J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

The result is

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

Since $D^{(j)}_{m'm}(\alpha,\beta,\gamma) = e^{-i(m'\alpha+m\gamma)}d^{(j)}_{m'm}(\beta)$, we finally obtain

$$D^{(1)}(\alpha,\beta,\gamma) = \begin{pmatrix} \frac{1}{2}e^{-i(\alpha+\gamma)}(1+\cos\beta) & -\frac{1}{\sqrt{2}}e^{-i\alpha}\sin\beta & \frac{1}{2}e^{-i(\alpha-\gamma)}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}e^{-i\gamma}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}e^{i\gamma}\sin\beta \\ \frac{1}{2}e^{i(\alpha-\gamma)}(1-\cos\beta) & \frac{1}{\sqrt{2}}e^{i\alpha}\sin\beta & \frac{1}{2}e^{i(\alpha+\gamma)}(1+\cos\beta) \end{pmatrix}$$

For fun, given this $D^{(1)}$-matrix, we can compute its character, as defined in problem 2.

$$\chi^{(1)}(\alpha,\beta,\gamma) = \operatorname{Tr} D^{(1)}(\alpha,\beta,\gamma) = \cos\beta + \cos(\alpha+\gamma)(1+\cos\beta)$$

This equation does not appear to have the same form as

$$\chi^{(1)}(\phi) = \frac{\sin\frac{3}{2}\phi}{\sin\frac{1}{2}\phi} = 1 + 2\cos\phi$$

However, this is because ϕ is a single rotation angle (about some $\hat{n}$ axis), whereas α, β, γ are a set of Euler angles. In fact, this allows us to solve for ϕ (the equivalent single rotation angle) in terms of the Euler angles

$$2\cos\phi = \cos\beta + \cos(\alpha+\gamma)(1+\cos\beta) - 1$$

This may be written more concisely as

$$\cos^2\frac{\phi}{2} = \cos^2\frac{\alpha+\gamma}{2}\cos^2\frac{\beta}{2}$$

[We would have obtained the square root of this relation had we looked at the $j=1/2$ character instead.]

4. The relation between spherical and cartesian components of a vector $\vec{V}$ is given by

$$V_0^{(1)} = V_z, \qquad V_{\pm1}^{(1)} = \mp\frac{1}{\sqrt{2}}(V_x \pm iV_y)$$

We may take a tensor product of two vectors $V_q^{(1)}$ and $W_q^{(1)}$ according to

$$T_q^{(k)} = \sum_{q_1,q_2} \langle 11; q_1 q_2 | 11; kq\rangle V_{q_1}^{(1)} W_{q_2}^{(1)}$$

[this is a special case of Sakurai (3.10.27)]

a) Show that $T_0^{(0)}$ is proportional to the scalar product $\vec{V}\cdot\vec{W}$.

This problem is an exercise in looking up Clebsch-Gordan coefficients. Since we are tensoring two vectors together, we need the table for **1** $\times$ **1**. *For $T_0^{(0)}$, we find*

$$\begin{aligned} T_0^{(0)} = \sum_{q=-1}^{1}\langle 11; q\ {-q}|11;00\rangle V_q W_{-q} &= \tfrac{1}{\sqrt{3}}(V_1W_{-1} - V_0W_0 + V_{-1}W_1) \\ &= -\tfrac{1}{\sqrt{3}}(V_xW_x + V_yW_y + V_zW_z) \\ &= -\tfrac{1}{\sqrt{3}}\vec{V}\cdot\vec{W} \end{aligned}$$

Note that Sakurai has a mistake of omitting the square root.

b) Show that $T_q^{(1)}$ is proportional to the (spherical tensor) components of $(\vec{V}\times\vec{W})$.

We calculate $T_1^{(1)}$, $T_0^{(1)}$ and $T_{-1}^{(1)}$ one at a time

$$
\begin{aligned}
T_1^{(1)} = \sum_{q=0}^{1}\langle 11; q\,1-q|11;11\rangle V_q W_{1-q} &= \tfrac{1}{\sqrt{2}}(V_1W_0 - V_0W_1)\\
&= -\tfrac{1}{2}\big((V_x+iV_y)W_z - V_z(W_x+iW_y)\big)\\
&= -\tfrac{i}{2}\big((V_yW_z - V_zW_y) + i(V_zW_x - V_xW_z)\big)\\
&= -\tfrac{i}{2}\big((\vec{V}\times\vec{W})_x + i(\vec{V}\times\vec{W})_y\big) = \tfrac{i}{\sqrt{2}}(\vec{V}\times\vec{W})_1\\
T_0^{(1)} = \sum_{q=-1}^{1}\langle 11; q\ -q|11;1\ -1\rangle V_q W_{-q} &= \tfrac{1}{\sqrt{2}}(V_1W_{-1} - V_1W_1)\\
&= -\tfrac{1}{2\sqrt{2}}\big((V_x+iV_y)(W_x-iW_y) - (V_x-iV_y)(W_x+iW_y)\big)\\
&= \tfrac{i}{\sqrt{2}}(V_xW_y - V_yW_x) = \tfrac{i}{\sqrt{2}}(\vec{V}\times\vec{W})_0\\
T_{-1}^{(1)} = \sum_{q=-1}^{0}\langle 11; q\ -1\ -q|11;1\,-1\rangle V_q W_{-1-q} &= \tfrac{1}{\sqrt{2}}(V_0W_{-1} - V_{-1}W_0)\\
&= \tfrac{1}{2}\big(V_z(W_x-iW_y) - (V_x-iV_y)W_z\big)\\
&= \tfrac{i}{2}\big((V_yW_z - V_zW_y) - i(V_zW_x - V_xW_z)\big)\\
&= \tfrac{i}{2}\big((\vec{V}\times\vec{W})_x - i(\vec{V}\times\vec{W})_y\big) = \tfrac{i}{\sqrt{2}}(\vec{V}\times\vec{W})_{-1}
\end{aligned}
$$

Putting this together gives simply

$$T_q^{(1)} = \tfrac{i}{\sqrt{2}}(\vec{V}\times\vec{W})_q$$

c) Write down the five components of $T_q^{(2)}$ in terms of $V_q^{(1)}$ and $W_q^{(1)}$.

According to the Clebsch-Gordan table, we have

$$
\begin{aligned}
T_2^{(2)} &= V_1W_1\\
T_1^{(2)} &= \tfrac{1}{\sqrt{2}}(V_1W_0 + V_0W_1)\\
T_0^{(2)} &= \tfrac{1}{\sqrt{6}}(V_1W_{-1} + 2V_0W_0 + V_{-1}W_1)\\
T_{-1}^{(2)} &= \tfrac{1}{\sqrt{2}}(V_0W_{-1} + V_{-1}W_0)\\
T_{-2}^{(2)} &= V_{-1}W_{-1}
\end{aligned}
$$

There is of course a certain symmetry in these terms for $q\leftrightarrow -q$. This is not a coincidence, but follows from the Clebsch-Gordan identity

$$\langle j_1j_2; -m_1\ -m_2|j_1j_2; j\ -m\rangle = (-)^{j_1+j_2+j}\langle j_1j_2; m_1m_2|j_1j_2; jm\rangle$$

d) Find $T_0^{(2)}$ in terms of the cartesian components V_x, V_y, V_z and W_x, W_y, W_z.

Using the relation between spherical and cartesian coordinates, we find

$$\begin{aligned} T_0^{(2)} &= \tfrac{1}{\sqrt{6}}\big(-\tfrac{1}{2}(V_x + iV_y)(W_x - iW_y) + 2V_z W_z - \tfrac{1}{2}(V_x - iV_y)(W_x + iW_y)\big) \\ &= \tfrac{1}{\sqrt{6}}(-V_x W_x - V_y W_y + 2V_z W_z) \end{aligned}$$

This may equivalently be written as

$$T_0^{(2)} = \tfrac{1}{\sqrt{6}}(3V_z W_z - \vec{V} \cdot \vec{W})$$

[Note that the answer is given by Sakurai (3.10.26), which incidentally has a typo in the first equation. However this problem is basically to prove those equations.]