**Physics 512** **Winter 2003**

## Homework Set #3 – Due Friday, January 31

1. Let $S^{(k_1)}_{q_1}$ and $T^{(k_2)}_{q_2}$ be two irreducible spherical tensor operators of ranks $k_1$ and $k_2$, respectively. We may form the tensor product $W^{(k)}_q$ defined by

$$W^{(k)}_q = \sum_{q_1+q_2=q} \langle k_1 k_2; q_1 q_2 | k_1 k_2; k\, q\rangle S^{(k_1)}_{q_1} T^{(k_2)}_{q_2}$$

Using the definition of spherical tensors in terms of their commutation properties

$$\begin{aligned} [J_z, T^{(k)}_q] &= q\hbar\, T^{(k)}_q \\ [J_\pm, T^{(k)}_q] &= \sqrt{k(k+1) - q(q\pm 1)}\,\hbar\, T^{(k)}_{q\pm 1} \end{aligned}$$

prove that $W^{(k)}_q$ is a spherical tensor operator of rank $k$.

2. The electric dipole operator is given by $\vec{d}_e = e\vec{r}$. Hence for an electric field in the $\hat{z}$ direction, electric dipole transitions for the Hydrogen atom (central force potential) are governed by the matrix element

$$\langle n', l', m' | z | n, l, m\rangle$$

*a*) What are the conditions on the quantum numbers for this matrix element to be non-vanishing? Make sure to note that the states are of definite parity.

*b*) Using the Wigner-Eckart theorem, write the matrix element

$$\langle n', l', m' | x | n, l, m\rangle$$

in terms of the above one (you do not have to evaluate the resulting Clebsch-Gordan coefficients). Give the selection rule on $m$ and $m'$.

3. The electric quadrupole moment operator is a spherical tensor of rank 2, defined as

$$Q^{(2)}_{\pm 2} = e\tfrac{\sqrt{6}}{2}(x \pm iy)^2, \qquad Q^{(2)}_{\pm 1} = \mp e\sqrt{6}(x\pm iy)z, \qquad Q^{(2)}_0 = e(3z^2 - r^2)$$

The actual quadrupole moment (expectation value) is defined as

$$\mathcal{Q} = \langle \alpha; j, j | Q^{(2)}_0 | \alpha; j, j\rangle$$

*a*) Show that $\mathcal{Q} = 0$ for $j = 0$ or $\frac{1}{2}$. This indicates that particles with spins less than one cannot have a quadrupole moment.

*b*) Using the fact that $T_0^{(2)} = 3J_z^2 - \vec{J}^{\,2}$ is the $q = 0$ component of a rank-2 tensor $T_q^{(2)}$, show that

$$\langle \alpha; j, m | Q_0^{(2)} | \alpha; j, m \rangle = \frac{3m^2 - j(j+1)}{j(2j-1)} \mathcal{Q}, \qquad \text{for } j \geq 1$$

Here $\vec{J}$ is the usual angular momentum operator.

*c*) Show that the expectation value of the operator $exz$ in a state $|\alpha; j, m\rangle$ vanishes.

4. Sakurai, Chapter 4, Problem 3. A quantum-mechanical state $|\Psi\rangle$ is known to be a simultaneous eigenstate of two Hermitian operators $A$ and $B$ which anticommute

$$AB + BA = 0$$

*a*) What can be said about the eigenvalues of $A$ and $B$ for state $|\Psi\rangle$?

*b*) For $A = U_P$ (the parity operator), and $B = \vec{p}$, what can be deduced about parity and momentum?