Physics 512 **Winter 2003**

Homework Set #6 – Due Monday, March 3

1. One dimensional scattering. Consider scattering from a potential $V(x) = -g\delta(x)$. If we send in a particle from the left

$$\psi_{\rm inc}(x) = e^{ikx} \qquad x < 0$$

there will be an amplitude for both reflection and transmission. The transmitted part may be written

$$\psi_{\rm trans} = S(E)e^{ikx} = \sqrt{T}e^{i(kx+\delta)} \qquad x > 0$$

where T is the (real) transmission coefficient and δ the phase shift (E is the energy).

a) Find the transmission coefficient and phase shift and show that T is insensitive to the sign of g. What are the limiting values of δ for $E \to 0$ and $E \to \infty$ (consider both positive and negative g)?

Although the incident wavefunction is e^{ikx}*, there must be a reflected one as well. Thus the complete wavefunction may be given as*

$$\psi(x) = \begin{cases} \psi_< = e^{ikx} + Be^{-ikx} & x < 0 \\ \psi_> = Se^{ikx} & x > 0 \end{cases}$$

We must satisfy the continuity and jump conditions at $x = 0$*:*

$$\psi_<(0) = \psi_>(0), \qquad \psi'_<(0) = \psi'_>(0) + \frac{2mg}{\hbar^2}\psi(0)$$

This gives a set of equations

$$1 + B = S, \qquad ik(1 - B) = ikS + \frac{2mg}{\hbar^2}S$$

which may be solved to give

$$S(E) = \left(1 - \frac{img}{k\hbar^2}\right)^{-1}, \qquad E = \frac{\hbar^2 k^2}{2m} \tag{1}$$

The transmission coefficient and phase shift is obtained by rewriting $S(E)$ *in terms of a magnitude and phase,* $S = \sqrt{T}\,e^{i\delta}$*. The result is simply*

$$T = \frac{1}{1 + \left(\frac{mg}{k\hbar^2}\right)^2}, \qquad \delta = \tan^{-1}\frac{mg}{k\hbar^2}$$

Since g *enters squared in* T*, the transmission coefficient is insensitive to the sign of* g*. However this is not the case for the phase shift. As for the limiting behavior of* δ*, we note that there is always a* 2π *phase ambiguity present. However we can* *define the phase to be 0 when $g \to 0$. This corresponds to taking $\tan^{-1}$ to lie between $-\pi/2$ and $\pi/2$. The limiting phase shifts are then given by*

	$E \to 0$	$E \to \infty$
$g > 0$	$\pi/2$	0
$g < 0$	$-\pi/2$	0

In general, we may understand the sign of the phase shift as being related to the attractive versus repulsive nature of the potential.

b) Show that $S(E)$ has a pole for imaginary k, corresponding to the bound state of the delta function potential. Show that this bound state only exists for $g > 0$ by considering the properties of the analytically continued scattering problem.

From (1), it is obvious that $S(E)$ has a pole at

$$k = i\frac{mg}{\hbar^2}$$

For $g > 0$ (corresponding to an attractive potential), this pole lies in the upper half of the complex k-plane. Since the transmitted wavefunction behaves as $\psi_{\text{trans}} \sim e^{ikx}$, we see that only complex values of k in the upper half plane leads to an exponentially decaying ψ_{trans}. Since we do not want the transmitted (nor reflected) wavefunction to grow exponentially without bound, this shows that the bound state can only exist for $g > 0$. Note, however, that the incident wavefunction does blow up at $x \to -\infty$. But, in some sense, since that is an incident wavefunction, we only care that it 'hits' the potential at $x = 0$ in a reasonable fashion and do not care about its behavior at $x \to -\infty$. [Actually, the bound state corresponds to having a transmitted and reflected wavefunction without any incident wavefunction at all].

2. Consider (classical) Rutherford scattering of a particle of mass m and incident energy E, with potential energy $V(r) = C/r$ (C constant).

a) Derive the relation

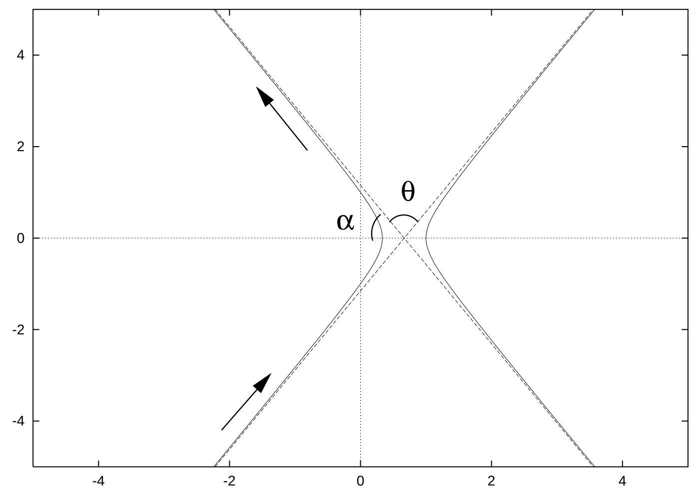
$$b = \frac{C}{2E}\cot\frac{\theta}{2}$$

between the impact parameter b and the scattering angle θ. [It may be useful to recall that Keplerian orbits can be written as

$$r(\theta) = \frac{L^2}{mC}(1 + e\cos\theta)^{-1}, \qquad e^2 = 1 + \frac{2EL^2}{mC^2}$$

(in polar coordinates, with energy E and angular momentum L) where the eccentricity $e > 1$ for hyperbolic trajectories.]

We use the orbital equation given above (although there are other equivalent ways to obtain the relation). Plotting $r(\theta)$ in polar coordinates, we see that the scattering geometry is given by



where the angle of the asymptote is given by $\cos\alpha = 1/e$. *The scattering angle* θ *is then related as* $\theta = \pi - 2\alpha$. *Hence*

$$e = \frac{1}{\cos\alpha} = \frac{1}{\cos\frac{\pi-\theta}{2}} = \frac{1}{\sin\frac{\theta}{2}}$$

We may also relate the impact parameter b *to angular momentum. Since* $L = bmv$, *we find* $L^2 = b^2m^2v^2 = 2mEb^2$. *Hence the eccentricity equation may be rewritten as*

$$e^2 = 1 + \left(\frac{2Eb}{C}\right)^2$$

or

$$b = \frac{C}{2E}\sqrt{e^2-1} = \frac{C}{2E}\sqrt{\csc^2\frac{\theta}{2} - 1} = \frac{C}{2E}\cot\frac{\theta}{2}$$

b) Derive the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{C^2}{16E^2}\frac{1}{\sin^4(\theta/2)}$$

[see Merzbacher Exercise 13.1] and show that the total cross section diverges.

We simply use

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{C^2}{4E^2}\frac{\cot\frac{\theta}{2}}{\sin\theta}\left|-\tfrac{1}{2}\csc^2\tfrac{\theta}{2}\right| = \frac{C^2}{16E^2}\frac{1}{\sin^4\frac{\theta}{2}}$$

The total cross section is given by integrating this over the solid angle, $\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega$. *However it is obvious that the differential cross section diverges in the forward direction* ($\theta \to 0$). *We may cut off the forward direction by demanding* $\theta_0 < \theta < \pi$. *Then*

$$\sigma_{\text{tot}} = 2\pi\int_{-1}^{\cos\theta_0}\frac{C^2}{16E^2}\left(\frac{2}{1-\cos\theta}\right)^2 d\cos\theta = \frac{\pi C^2}{2E^2}\int_{-1}^{\cos\theta_0}\frac{dx}{(1-x)^2}$$
$$= \frac{\pi C^2}{2E^2}\left(\frac{1}{1-\cos\theta_0} - \frac{1}{2}\right) = \frac{\pi C^2}{4E^2}\cot^2\frac{\theta_0}{2} \approx \frac{\pi C^2}{E^2\theta_0^2}$$

Note that $\sigma_{\text{tot}} \approx \pi b_0^2$ where b_0 is the impact parameter corresponding to the cutoff scattering angle θ_0. This is just the cross sectional area given by geometry.

3. Merzbacher, Exercise 13.7 *and* Problem 13.2. Scattering from a translationally invariant potential.

   We start with Exercise 13.7.

   a) *Here it is convenient to consider the integral equation*

$$\psi_{\vec{k}}^{(\pm)}(\vec{r}) = Ne^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int G(\vec{r},\vec{r}\,')V(\vec{r}\,')\psi_{\vec{k}}^{(\pm)}(\vec{r}\,')\,d^3\vec{r}\,' \tag{2}$$

   Letting $\vec{r} \to \vec{r} + \vec{R}$, this becomes

$$\psi_{\vec{k}}^{(\pm)}(\vec{r}+\vec{R}\,) = e^{i\vec{k}\cdot\vec{R}}Ne^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int G(\vec{r}+\vec{R},\vec{r}\,')V(\vec{r}\,')\psi_{\vec{k}}^{(\pm)}(\vec{r}\,')\,d^3\vec{r}\,'$$

   However, since $\vec{r}\,'$ is an integration variable, we may shift $\vec{r}\,' \to \vec{r}\,' + \vec{R}$. Noting the periodicity of the potential, $V(\vec{r}+\vec{R}\,) = V(\vec{r}\,)$, and the translational invariance of the Green's function, $G(\vec{r}+\vec{R},\vec{r}\,'+\vec{R}\,) = G(\vec{r},\vec{r}\,')$, we obtain

$$\psi_{\vec{k}}^{(\pm)}(\vec{r}+\vec{R}\,) = e^{i\vec{k}\cdot\vec{R}}Ne^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int G(\vec{r},\vec{r}\,')V(\vec{r}\,')\psi_{\vec{k}}^{(\pm)}(\vec{r}\,'+\vec{R}\,)\,d^3\vec{r}\,'$$

   or

$$\left[e^{-i\vec{k}\cdot\vec{R}}\psi_{\vec{k}}^{(\pm)}(\vec{r}+\vec{R}\,)\right] = Ne^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int G(\vec{r},\vec{r}\,')V(\vec{r}\,')\left[e^{-i\vec{k}\cdot\vec{R}}\psi_{\vec{k}}^{(\pm)}(\vec{r}\,'+\vec{R}\,)\right]d^3\vec{r}\,'$$

   This demonstrates that the function $e^{-i\vec{k}\cdot\vec{R}}\psi_{\vec{k}}^{(\pm)}(\vec{r}+\vec{R}\,)$ satisfies the same integral equation, (2), as $\psi(\vec{r}\,)$ itself. Assuming the solution to the integral equation is unique (which it should be, at least from physical grounds) this demonstrates that

$$\psi_{\vec{k}}^{(\pm)}(\vec{r}+\vec{R}\,) = e^{i\vec{k}\cdot\vec{R}}\psi_{\vec{k}}^{(\pm)}(\vec{r}\,)$$

   so that $\psi_{\vec{k}}^{(\pm)}$ are Bloch wavefunctions. Of course, to be complete, we really ought to prove the uniqueness of the solution. But that is a bit more involved.

   b) *For the scattering amplitude, we note that*

$$f_{\vec{k}}(\hat{r}) = -\frac{m}{2\pi\hbar^2 N}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'}V(\vec{r}\,')\psi_{\vec{k}}^{(+)}(\vec{r}\,')\,d^3\vec{r}\,'$$

   Once again, we shift the integration variable, $\vec{r}\,' \to \vec{r}\,' + \vec{R}$, to get

$$\begin{aligned} f_{\vec{k}}(\hat{r}) &= -\frac{m}{2\pi\hbar^2 N}\int e^{-i\vec{k}\,'\cdot\vec{R}}e^{-i\vec{k}\,'\cdot\vec{r}\,'}V(\vec{r}\,')\psi_{\vec{k}}^{(+)}(\vec{r}\,'+\vec{R})\,d^3\vec{r}\,' \\ &= -e^{i(\vec{k}-\vec{k}\,')\cdot\vec{R}}\frac{m}{2\pi\hbar^2 N}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'}V(\vec{r}\,')\psi_{\vec{k}}^{(+)}(\vec{r}\,')\,d^3\vec{r}\,' \\ &= e^{i(\vec{k}-\vec{k}\,')\cdot\vec{R}}f_{\vec{k}}(\hat{r}) \end{aligned}$$

Hence $(1-e^{i(\vec{k}-\vec{k}\,')\cdot\vec{R}})f_{\vec{k}}(\hat{r})=0$*, or* $f_{\vec{k}}(\hat{r})=0$ *unless* $e^{i(\vec{k}-\vec{k}\,')\cdot\vec{R}}=1$*, corresponding to the Laue condition,* $(\vec{k}-\vec{k}\,')\cdot\vec{R}=2\pi n$*.*

We now consider Problem 13.2 for scattering with a translationally invariant potential in the Born approximation. In this case, the Born amplitude is

$$\begin{aligned} f_k(\hat{r}) &= -\frac{m}{2\pi\hbar^2}\int e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}V(\vec{r})\,d^3\vec{r} \\ &= -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\,d^3\vec{r} \end{aligned}$$

where $\vec{q}=\vec{k}'-\vec{k}$ *is the momentum transfer. Since the potential is translationally invariant, and since* $\vec{r}$ *is just an integration variable, we may shift* $\vec{r}\to\vec{r}+\vec{R}$ *inside the integral. This gives*

$$\begin{aligned} f_k(\hat{r}) &= -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{R}}e^{-i\vec{q}\cdot\vec{r}}V(\vec{r}+\vec{R})\,d^3\vec{r} \\ &= e^{-i\vec{q}\cdot\vec{R}}\left(-\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\,d^3\vec{r}\right) \\ &= e^{-i\vec{q}\cdot\vec{R}}f_k(\hat{r}) \end{aligned}$$

Once again, this leads to the condition $(1-e^{-i\vec{q}\cdot\vec{R}})f_k(\hat{r})=0$*, indicating that scattering vanishes unless* $\vec{q}\cdot\vec{R}=2\pi n$*. [This must be one of the simplest problems in Merzbacher!]*