Physics 512 **Winter 2003**

Homework Set #7 – Solutions

1. Consider the scattering of a beam of spinless particles of momentum $\hbar k$ initially traveling along the $+\hat{z}$ direction by a potential of the form

$$V(\vec{r}) = V_0\big[\delta(x)\delta(y-b)\delta(z) + \delta(x)\delta(y+b)\delta(z)\big]$$

a) Calculate the scattering amplitude and differential cross section in the Born approximation.

Using the Born approximation, we compute

$$\begin{aligned} f_{\text{Born}}(\vec{q}) &= -\frac{m}{2\pi\hbar^2}\int V(\vec{r}\,')e^{-i\vec{q}\cdot\vec{r}\,'}\,d^3\vec{r}\,' \\ &= -\frac{mV_0}{2\pi\hbar^2}\big[e^{-iq_yb} + e^{iq_yb}\big] = -\frac{mV_0}{\pi\hbar^2}\cos q_yb \end{aligned}$$

where $\vec{q} = \vec{k}\,' - \vec{k}$. *Since the beam initially travels in the* $+\hat{z}$ *direction, we must have* $k_y = 0$. *Hence we may write*

$$f_{\text{Born}}(\vec{q}) = -\frac{mV_0}{\pi\hbar^2}\cos k'_yb$$

so that

$$\frac{d\sigma}{d\Omega} = |f_{\text{Born}}|^2 = \left(\frac{mV_0}{\pi\hbar^2}\right)^2\cos^2 k'_yb$$

Using spherical coordinates and conservation of energy ($|\vec{k}\,'| = |\vec{k}\,|$) *this may be written as*

$$\frac{d\sigma}{d\Omega} = \left(\frac{mV_0}{\pi\hbar^2}\right)^2\cos^2\big(kb\sin\theta\sin\varphi\big)$$

b) This potential represents scatterers located at $y = -b$ and $y = +b$. How does the quantum result differ from what one would expect classically?

The quantum scattering cross section has the form

$$\frac{d\sigma}{d\Omega} \sim \cos^2(\textit{scattering direction})$$

so it results in an interference pattern (very similar to that of a double slit). Classically, particles would scatter from either the first scatterer or the second, but not both simultaneously (multiple scattering could occur, but only sequentially). Hence there would be no interference classically (which sounds pretty obvious when one thinks about it).

2. Merzbacher, Exercise 13.11. If $V = C/r^n$, obtain the functional dependence of the Born scattering amplitude on the scattering angle. Discuss the reasonableness of the result qualitatively. What values of n give a meaningful answer?

   Since the potential is spherically symmetric, we may use the expression

$$f_{\text{Born}}(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty V(r)\frac{\sin qr}{qr}\,r^2\,dr$$
$$= -\frac{2mC}{q\hbar^2}\int_0^\infty \frac{\sin qr}{r^{n-1}}\,dr$$

   where $q = 2k\sin\frac{\theta}{2}$. Before attempting to integrate this, we may obtain the functional dependence of this expression on q by simply performing a change of variables, $x = qr$. The result is

$$f_{\text{Born}}(\theta) = -\frac{2mC}{\hbar^2}I_n q^{n-3} = -\frac{2mC}{\hbar^2}I_n(2k)^{n-3}\sin^{n-3}\frac{\theta}{2}$$

   where

$$I_n = \int_0^\infty \frac{\sin x}{x^{n-1}}\,dx \tag{1}$$

   Ignoring for the moment the convergence of I_n, we find the general behavior

$$f_{\text{Born}}(\theta) \sim \sin^{n-3}\frac{\theta}{2}$$

   Note that we assume n is non-negative (but not necessarily to be an integer), otherwise the potential would grow indefinitely for large r, in violation of the scattering assumptions. For $n < 3$, this blows up in the forward direction, corresponding to enhanced small angle scattering. However there is an opposite effect of suppression for $n > 3$. This at least seems reasonable, since the larger n is, the steeper (and more highly localized) the potential is. For a very sharp potential, most of the time the incoming particles would miss it altogether. But in the instances when they hit the potential, they would most likely get scattered by large angles.

   There is, however, a question of convergence of I_n. Examining (1), we see that this integral may blow up either as $x \to 0$ or $x \to \infty$. At the lower limit, $\sin x \approx x$, so the behavior is governed by

$$\int_0 \frac{sinx}{x^{n-1}}\,dx \sim \int_0 x^{2-n}\,dx$$

   Convergence demands $2 - n > -1$ or $n < 3$. For the upper limit, on the other hand, $\sin x$ is oscillatory, and as long as the oscillatory integrand is decreasing, it would be expected to converge. This yields the condition $n - 1 > 0$ or $n > 1$. Thus I_n only converges for n in the range $1 < n < 3$. Note that the Coulomb potential has $n = 1$ exactly, and is somewhat delicate to treat. (As we have seen, for this case one could introduce a Yukawa like factor to make the integral well defined). Including the

Coulomb case, this suggests that n in the range $1 \leq n < 3$ will give a meaningful answer.

In reality, the situation is quite a bit more complicated. For example, for $1 < n < 3$ (ie the convergent case), the integral (1) may actually be evaluated to yield

$$I_n = \Gamma(2-n)\sin\frac{n\pi}{2}$$

In this case, one may be tempted to use analytic continuation to define I_n for all values of n (positive or negative). One would then conclude that the only dangerous values of n are those that give rise to poles in I_n, namely $n = 3, 5, 7, \ldots$. However this cannot be the complete answer. Clearly, as indicated above, all negative values of n must be discarded since they correspond to a growing potential, which cannot allow scattering states. In addition, note that the Born approximation does not distinguish between positive and negative potentials. However imagine having an attractive $n > 2$ potential. In the radial Schrödinger equation, the effective potential then has the form

$$V_{\text{eff}}(r) = -\frac{|C|}{r^n} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

and the first term would always dominate the angular momentum barrier. Thus any particle sent into this potential would get captured and essentially absorbed by the system. Actually the $\ell = 0$ case has to be considered separately. But for this case, one can appeal to the $\Delta x \Delta p_x$ uncertainty relation and show that it still allows s-wave particles to be captured. In general, absorption of particles cannot be described by a self-adjoint Hamiltonian, and this happens for $n > 2$ (even though formally it looks okay). The $n = 2$ case is interesting since it is just at the borderline of being ill behaved. There is a specialized field of the study of such problems, including the construction of self-adjoint extensions of seemingly ill-behaved Hamiltonians. For example, given the potential $V = a/r^2$, it is known that the problem is well posed for $a \geq -\frac{1}{4}(\hbar^2/2m)$. However self-adjoint extensions do exist for more negative values of a.

3. This is based on Sakurai, Chapter 7, Problem 3. [See also Merzbacher Eq. (13.85) and Exercise 13.16; however you should actually work out the relevant partial wave phase shifts.] Consider a potential

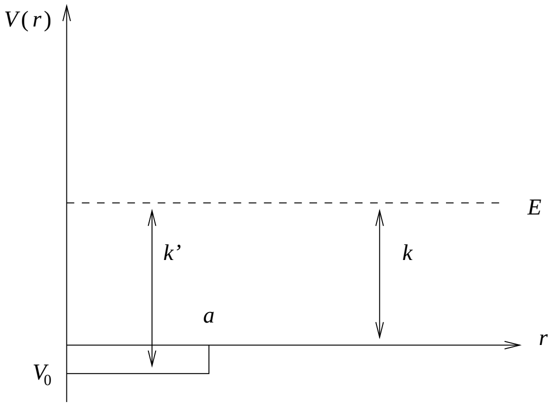
$$V = \begin{cases} V_0, & \text{for } r < a; \\ 0, & \text{for } r > a \end{cases}$$

where V_0 is constant and may be positive or negative.

a) For $|V_0| \ll E = \hbar^2k^2/2m$ and $ka \ll 1$, the differential cross section is dominated by the s-wave phase shift. Using the method of partial waves, show that in this limit the total cross section is given by

$$\sigma_{\text{tot}} = \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}$$

Instead of simply quoting the answer from Merzbacher, we may calculate the cross section using the appropriate limits from the beginning. The basic setup is as follows



where $\hbar k = \sqrt{2mE}$ *and* $\hbar k' = \sqrt{2m(E - V_0)}$. *Since* $|V_0| \ll E$, *we see that both* $ka \ll 1$ *and* $k'a \ll 1$. *This ensures that only s-wave* ($\ell = 0$) *scattering is important (so that the scattering will be isotropic). To obtain the cross section, we first solve the Schrödinger equation for* $r < a$ *and then match it to the outside solution through the logarithmic derivative* β_ℓ. *Once we find* β_ℓ, *we then substitute it in the expression for* $\cot\delta_\ell$ *and thus arrive at the cross section.*

For $r < a$, *the radial wavefunction reaches the origin. Hence only the Bessel function* j_0 *is allowed. Since*

$$R(r) = j_0(k'r) = \frac{\sin k'r}{k'r} = 1 - \tfrac{1}{6}(k'r)^2 + \cdots$$

we easily obtain

$$\beta_0 = k'a\frac{j_0'(k'a)}{j_0(k'a)} = -\tfrac{1}{3}(k'a)^2 + \cdots$$

Note that we need to keep the second term in the Taylor expansion of sin *to get a non-vanishing result. We now recall that*

$$\cot\delta_0 = \frac{ka\,n_0'(ka) - \beta_0 n_0(ka)}{ka\,j_0'(ka) - \beta_0 j_0(ka)}$$

Using again $j_0(z) = \sin(z)/z = 1 - \frac{1}{6}z^2 + \cdots$ *and* $n_0(z) = -\cos(z)/z = -1/z + \cdots$, *we find*

$$\cot\delta_0 \approx \frac{1/(ka)}{-\frac{1}{3}(ka)^2 - \beta_0} = -\frac{3}{ka[(ka)^2 + 3\beta_0]}$$

where we have used the fact that $\beta_0 \sim (k'a)^2 \sim (ka)^2 \ll 1$ *to decide which terms are important to keep in the numerator and denominator. Substituting in* $\beta_0 = -\frac{1}{3}(k'a)^2$ *and noting that* $k^2 - k'^2 = 2mV_0/\hbar^2$ *then gives*

$$\cot\delta_0 \approx -\frac{3}{ka[(ka)^2 - (k'a)^2]} = -\frac{3\hbar^2}{2mkV_0a^3} = -3\left(\frac{\hbar^2k^2}{2mV_0}\right)\frac{1}{(ka)^3}$$

Since $ka \ll 1$ *and* $2mV_0/\hbar^2k^2 \ll 1$, $\cot\delta_0$ *is very large (and negative). To a good approximation, we may write*

$$\delta_0 \approx -\frac{2mV_0a^2}{3\hbar^2}(ka) \tag{2}$$

The $\ell = 0$ *cross section is then*

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 = \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}$$

This could have been obtained from Merzbacher (13.85) by expanding $\tan(x)/x = 1 + \frac{1}{3}x^2 + \cdots$, *although one may worry about the order of limits (ie* $E \to 0$ *versus* $V_0 \to 0$*).*

b) Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta$$

Obtain an approximate expression for B/A.

If the energy is raised slightly, then the next partial wave ($\ell = 1$*) will start to contribute. Since p-wave scattering is governed by the Legendre polynomial* $P_1(\cos\theta) = \cos\theta$, *this is where the purported angular distribution comes from. To see this, we note that*

$$f_k(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)f_\ell(k) \approx f_0(k) + 3f_1(k)\cos\theta$$

where

$$f_\ell(k) = \frac{1}{k(\cot\delta_\ell - i)}$$

is the ℓ*-th partial wave amplitude. Then*

$$\begin{aligned}\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 \approx |f_0(k) + 3f_1(k)\cos\theta|^2 &= |f_0|^2 + 6\,\mathrm{Re}\,(f_0^*f_1)\cos\theta + 9|f_1|^2\cos^2\theta \\ &\approx |f_0|^2 + 6\,\mathrm{Re}\,(f_0^*f_1)\cos\theta\end{aligned}$$

(assuming $|f_1| \ll |f_0|$*). This allows us to identify*

$$A = |f_0|^2, \qquad B = 6\,\mathrm{Re}\,(f_0^*f_1) \tag{3}$$

The s-wave amplitude was already obtained in a) above

$$f_0 = \frac{1}{k(\cot\delta_0 - i)} \approx \frac{\delta_0}{k} \approx -\frac{2mV_0a^3}{3\hbar^2}$$

To obtain the p-wave amplitude, we once again start with the $r < a$ solution (but this time with $\ell = 1$). First note the series expansions

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} = \tfrac{1}{3}z - \tfrac{1}{30}z^3 + \cdots$$
$$n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} = -1/z^2 + \cdots$$

Thus the inside solution is given by

$$R(r) = j_1(k'r) = \tfrac{1}{3}k'r - \tfrac{1}{30}(k'r)^3 + \cdots$$

As a result

$$\beta_1 = k'a\frac{j_1'(k'a)}{j_1(k'a)} \approx k'a\frac{\frac{1}{3} - \frac{1}{10}(k'a)^2}{\frac{1}{3}k'a - \frac{1}{30}(k'a)^3} = 1 - \tfrac{1}{5}(k'a)^2 + \cdots$$

(notice that, in contrast to β_0, this expression starts with 1; this is because the radial wavefunction vanishes at $r = 0$ for $\ell > 0$). Next we find

$$\begin{aligned}\cot\delta_1 &= \frac{ka\,n_1'(ka) - \beta_1 n_1(ka)}{ka\,j_1'(ka) - \beta_1 j_1(ka)}\\ &\approx \frac{2/(ka)^2 + 1/(ka)^2}{\frac{1}{3}ka(1 - \frac{3}{10}(ka)^2) - \frac{1}{3}ka(1 - \frac{1}{10}(ka)^2 - \frac{1}{5}(k'a)^2)}\\ &= -\frac{45}{(ka)^3[(ka)^2 - (k'a)^2]} = -\frac{45\hbar^2}{2mk^3V_0a^5} = -45\left(\frac{\hbar^2k^2}{2mV_0}\right)\frac{1}{(ka)^5}\end{aligned}$$

As expected, δ_1 is very small, so we may write

$$\delta_1 \approx -\frac{2mV_0a^2}{45\hbar^2}(ka)^3$$

This may be compared with δ_0 given in (2). In particular, we see the expected $\delta_\ell \sim (ka)^{(2\ell+1)}$ behavior. Finally, we obtain

$$f_1 = \frac{1}{k(\cot\delta_1 - i)} \approx \frac{\delta_1}{k} \approx -\frac{2mV_0a^3}{45\hbar^2}(ka)^2 \approx \frac{f_0}{15}(ka)^2$$

Since both f_0 and f_1 are real, from (3) we have simply

$$B = 6A\frac{f_1}{f_0} \approx \tfrac{2}{5}(ka)^2A$$

Thus $B/A \approx \frac{2}{5}(ka)^2$, and the differential cross section is

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{4}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}\left(1 + \frac{2}{5}(ka)^2\cos\theta\right) \tag{4}$$

Note that in the limit $E \to 0$ (corresponding to $k \to 0$), the $\cos\theta$ *term drops out, and we get back the isotropic cross section found in a).*

4. This is essentially Merzbacher, Exercise 13.9. Compute the differential cross section for the potential of problem 3 using the Born approximation.

   a) Show that the differential cross section agrees with the result of problem 3*b*) in the limit $ka \ll 1$.

   For the potential $V(r) = V_0$, $(r < a)$, the Born approximation gives simply

   $$f_{\text{Born}}(\theta) = -\frac{2mV_0}{\hbar^2}\int_0^a \frac{\sin qr}{qr} r^2\,dr$$
   $$= -\frac{2mV_0}{\hbar^2 q^3}\int_0^{qa} x\sin x\,dx = -\frac{2mV_0}{\hbar^2 q^3}(\sin qa - qa\cos qa)$$

   where $q = 2k\sin\frac{\theta}{2} = k\sqrt{2(1-\cos\theta)}$. Hence the differential cross section is

   $$\frac{d\sigma}{d\Omega} = |f_{\text{Born}}(\theta)|^2 = \frac{4m^2V_0^2}{\hbar^4 q^6}(\sin qa - qa\cos qa)^2 \tag{5}$$

   We may now expand this for $ka \ll 1$ (which also corresponds to $qa \ll 1$. Since

   $$\sin x - x\cos x \approx (x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots) - x(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots) \approx \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5$$

   we obtain

   $$\frac{d\sigma}{d\Omega} \approx \frac{4m^2V_0^2}{9\hbar^4}a^6\left(1 - \frac{1}{5}(qa)^2\right) = \left(\frac{4}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}\left(1 - \frac{2}{5}(ka)^2(1-\cos\theta)\right)$$
   $$\approx \left(\frac{4}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}\left(1 + \frac{2}{5}(ka)^2\cos\theta\right)$$

   Note that in the second line we have dropped $\frac{2}{5}(ka)^2$ compared to 1. This agrees with the expression (4) found in problem 3b).

   b) Evaluate the total cross section in the Born approximation (for any value of ka).

   To evaluate the total cross section, we need to integrate (5) over the solid angle

   $$\sigma_{\text{tot}} = \frac{4m^2V_0^2}{\hbar^4}\int \frac{(\sin qa - qa\cos qa)^2}{q^6}\,d\varphi\,d\cos\theta$$
   $$= \frac{8\pi m^2V_0^2}{\hbar^4}\int_{-1}^{1}\frac{(\sin qa - qa\cos qa)^2}{q^6}\,d\cos\theta$$

   (where we need to keep in mind that q depends on θ). It is convenient to perform a change of variables

   $$x = qa = ka\sqrt{2(1-\cos\theta)}, \quad \textit{or} \quad \cos\theta = 1 - \frac{x^2}{2(ka)^2}$$

Since

$$d\cos\theta = -\frac{x}{(ka)^2}\,dx$$

we find

$$\begin{aligned}\sigma_{\rm tot} &= \frac{8\pi m^2V_0^2a^6}{\hbar^4}\int_0^{2ka}\frac{(\sin x - x\cos x)^2}{x^6}\frac{x}{(ka)^2}dx\\ &= \frac{8\pi m^2V_0^2a^6}{\hbar^4(ka)^2}\int_0^{2ka}\left(\frac{\sin^2 x}{x^5}-\frac{2\sin x\cos x}{x^4}+\frac{\cos^2 x}{x^3}\right)dx\end{aligned}$$

When the integrand is expanded as in the second line, each individual term is divergent at the lower limit $x=0$. However this divergence cancels when the terms are added together. Since this lower limit is slightly delicate, we may instead integrate from ϵ to $2ka$, and then take the limit $\epsilon\to 0$. For the regulated integral, we may integrate the first term by parts to obtain

$$\sigma_{\rm tot} = \frac{8\pi m^2V_0^2a^6}{\hbar^4(ka)^2}\left[-\frac{\sin^2 x}{4x^4}\bigg|_\epsilon^{2ka}+\int_\epsilon^{2ka}\left(-\frac{3\sin x\cos x}{2x^4}+\frac{\cos^2 x}{x^3}\right)dx\right]$$

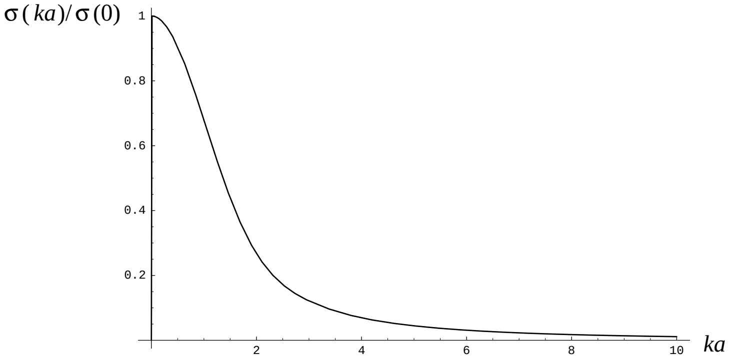
One more integration by parts then yields

$$\begin{aligned}\sigma_{\rm tot} &= \frac{8\pi m^2V_0^2a^6}{\hbar^4(ka)^2}\left[-\frac{\sin^2 x}{4x^4}+\frac{\sin x\cos x}{2x^3}\bigg|_\epsilon^{2ka}+\int_\epsilon^{2ka}\frac{1}{2x^3}\,dx\right]\\ &= \frac{8\pi m^2V_0^2a^6}{\hbar^4(ka)^2}\left[-\frac{\sin^2 x}{4x^4}+\frac{\sin x\cos x}{2x^3}-\frac{1}{4x^2}\right]_\epsilon^{2ka}\\ &= \frac{8\pi m^2V_0^2a^6}{\hbar^4(ka)^2}\left[-\frac{\sin^2 2ka}{4(2ka)^4}+\frac{\sin 2ka\cos 2ka}{2(2ka)^3}-\frac{1}{4(2ka)^2}+\frac{1}{4}\right]\end{aligned}$$

Note that the factor $1/4$ arises from the $\epsilon\to 0$ limit. This result may be rewritten as

$$\begin{aligned}\sigma_{\rm tot} &= \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}\times\frac{9(-\sin^2 2ka+4ka\sin 2ka\cos 2ka-(2ka)^2+(2ka)^4)}{2(2ka)^6}\\ &\approx \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2a^6}{\hbar^4}\left(1-\frac{2}{5}(ka)^2+\cdots\right)\end{aligned}$$

Despite the trig functions, a plot of $\sigma_{\rm tot}$ versus ka looks like



which does not have any wiggles.