Physics 512 **Winter 2003**

Homework Set #8 – Solutions

1. This is based on Sakurai, Chapter 7, Problem 9. Consider scattering by a repulsive δ-shell potential
$$\frac{2mV(r)}{\hbar^2} = \gamma\,\delta(r-a), \qquad \gamma > 0$$

   a) Set up an equation that determines the s-wave phase shift, δ_0, as a function of k (where $E = \hbar^2 k^2/2m$).

   Because of the δ-function, we expect the first derivative of the wavefunction to jump at $r = a$. For this reason, we cannot use the standard logarithmic derivative matching conditions. However this problem is easy to solve for the $\ell = 0$ phase shift. For $r \neq a$, all we have are free spherical waves. So we may write the exact solution for the radial wavefunction as
$$R(r) = \begin{cases} j_0(kr), & r < a \\ B[h_0^{(2)}(kr) + e^{2i\delta_0} h_0^{(1)}(kr)], & r \geq a \end{cases} \tag{1}$$
   where B is a normalization constant, and δ_0 is the s-wave phase shift. Actually, for the δ-function jump, it is better to work with the function $u(r) = rR(r)$, which satisfies the simple one-dimensional equation
$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2mV}{\hbar^2} + k^2\right] u(r) = 0$$
   For $\ell = 0$ and the δ-shell potential, this becomes simply
$$u'' + k^2 u = \gamma\delta(r-a)u \tag{2}$$
   In terms of $u(r)$, the radial wavefunction (1) becomes
$$u(r) = \begin{cases} k^{-1}\sin(kr), & r < a \\ ik^{-1}B[e^{-ikr} - e^{2i\delta_0} e^{ikr}], & r \geq a \end{cases}$$
   For convenience, we may choose a different normalization in order to eliminate the k^{-1} factor. Thus we write
$$u(r) = \begin{cases} \sin(kr), & r < a \\ B\sin(kr + \delta_0), & r \geq a \end{cases}$$
   (where we have additionally absorbed a phase $e^{i\delta_0}$ into B). This clearly solves (2) for $r \neq a$, and moreover retains the correct definition of the phase shift δ_0. From (2), we now match the wavefunction and first derivative at $r = a$. The two conditions give
$$B\sin(ka + \delta_0) = \sin(ka)$$
$$kB\cos(ka + \delta_0) = k\cos(ka) + \gamma\sin(ka)$$

Dividing the two eliminates B, and yields the result

$$\cot(ka + \delta_0) = \cot(ka) + \frac{\gamma}{k}$$

Using the relation $\cot(\alpha+\beta) = (\cot\alpha\cot\beta - 1)/(\cot\alpha + \cot\beta)$*, and solving for* $\cot\delta_0$*, we find the equivalent condition*

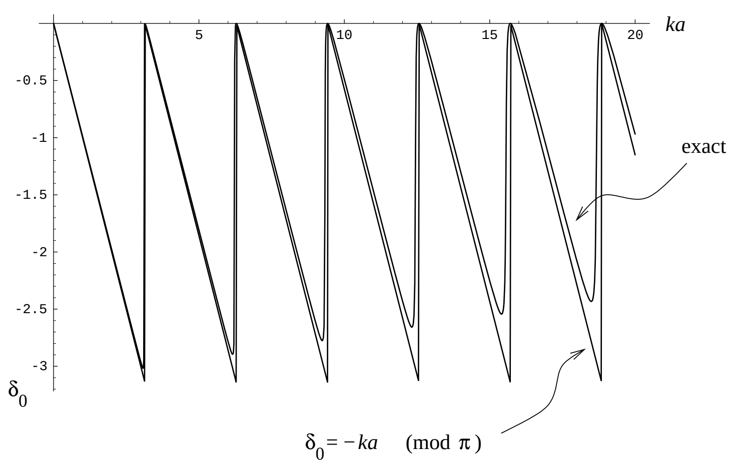
$$\cot\delta_0 = -\cot(ka) - \frac{k}{\gamma}\csc^2(ka) \tag{3}$$

b) Assume that γ is very large

$$\gamma \gg \frac{1}{a}, \qquad \gamma \gg k$$

Show that if $\tan ka$ is not close to zero, the s-wave phase shift resembles the hard-sphere result, $\delta_0 = -ka$.

Since $\tan ka$ *is not close to zero,* $\sin ka$ *is not close to zero, and* $\csc^2(ka)$ *is bounded. Thus for* $\gamma \gg k$*, we may drop the last term in* (3) *and find* $\cot\delta_0 \approx -\cot(ka)$*. This demonstrates that* $\delta_0 \approx -ka$*, which is the hard sphere result. Note that this result does not depend on ka being small, (although it does depend on the condition* $ka \ll \gamma a$ *being satisfied). A comparison of the exact result with the hard sphere approximation is shown by*



Note, of course, that the phase shift is only defined up to factors of $n\pi$*. This shows that, for the most part, the spherical shell behaves just like a hard sphere. However part c) below shows us that it is not just a hard sphere...*

c) Also show that for $\tan ka$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot\delta_0$ goes through zero from the positive side as k increases (so that δ_0 is increasing as k is increasing). Determine approximately the positions of the resonances keeping terms of order $1/\gamma$.

Since the s-wave cross section is given by

$$\sigma_0 = \frac{4\pi}{k^2}\frac{1}{\cot^2\delta_0 + 1}$$

we see that peaks in the cross section occur whenever $\cot\delta_0 = 0$*. Using* (3)*, this condition gives* $\cot(ka) = -(k/\gamma)\csc^2(ka)$ *or* $\sin(ka)\cos(ka) = -k/\gamma$*, which is the same as* $\sin(2ka) = -2k/\gamma$*. For* $\gamma \gg k$*, this shows that* $\sin(2ka) \approx 0$*, so that* $ka \approx n\pi/2$*. For an improved approximation, we let* $ka = n\pi/2 + \epsilon$ *and expand for small* ϵ *to find*

$$(-)^n \sin(2\epsilon) = -\frac{n\pi + 2\epsilon}{\gamma a} \quad \Rightarrow \quad 2\epsilon \approx (-)^{n+1}\frac{n\pi}{\gamma a} \quad \Rightarrow \quad \epsilon \approx (-)^{n+1}\frac{n\pi}{2\gamma a}$$

Thus we have

$$ka \approx \frac{n\pi}{2}\left(1 + (-)^{n+1}\frac{1}{\gamma a}\right) \tag{4}$$

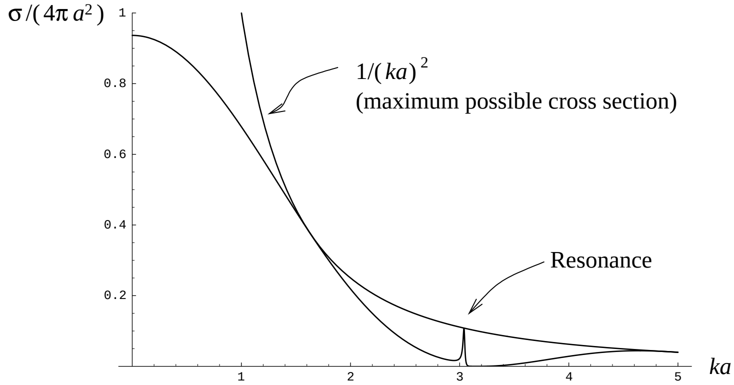
However care must be taken in that not all peaks are resonances. To have a resonance, the phase shift δ_0 *must be increasing as the energy is increasing. One way to see this is to recall the connection between attractive potentials and positive phase shifts. In a way, a positive phase shift corresponds to a delay in the particle's motion caused by the attractive potential. Since a resonance can be thought of as a quasi-bound state (where the particle bounces around a bit in the potential before coming out), we expect the phase shift to increase as we approach the resonant energy from below. We now reexamine* (4)*. For* $ka \approx n\pi/2$*, either* $\cot(ka)$ *or* $\tan(ka)$ *is close to zero. In the former case, the result of* b) *indicates that* $\delta_0 \approx -ka$ *is decreasing, so that there is no resonance. For* $\tan(ka)$ *close to zero, on the other hand, we rewrite* (4) *to give*

$$\begin{aligned}\cot\delta_0 &= -\frac{k}{\gamma}\cot^2(ka)\left[\tan^2(ka) + \frac{\gamma}{k}\tan(ka) + 1\right] \\ &\approx -\cot^2(ka)\left[\tan(ka) + \frac{k}{\gamma}\right]\end{aligned} \tag{5}$$

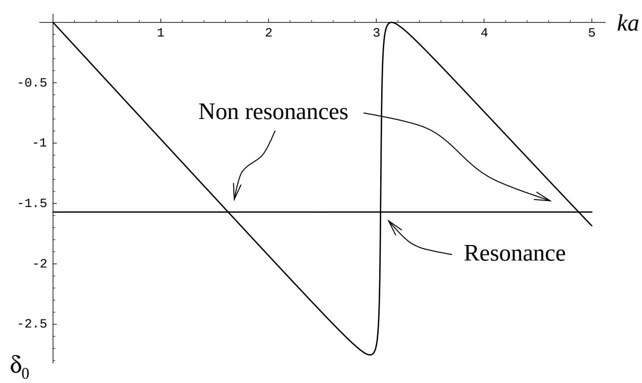
For increasing k*, the term in the brackets is increasing. Multiplied by a negative factor, this gives a decreasing* $\cot\delta_0$ *(which in turn yields an increasing* δ_0*). Therefore in this case, we have a true resonance. We can, in fact, solve* $\cot\delta_0 = 0$ *using the approximation in* (5) *to arrive at*

$$ka \approx n\pi - \frac{n\pi}{\gamma a} = n\pi\left(1 - \frac{1}{\gamma a}\right) \qquad (n > 0)$$

which agrees with (4)*. This equation gives the approximate positions of the resonances. As an example, for* $\gamma a = 30$*, we find a first resonance at* $ka \approx \pi(1 - 1/30) = 3.03687$*, which is close to the actual value of* $ka = 3.03957$*. The resonant behavior of the cross section looks like*



while the phase shift looks like



In particular, we see that while $\cot\delta_0$ *vanishes for* $ka \approx \pi/2$ *and* $3\pi/2$ *(and the cross section attains its maximum at these values), the phase shift is decreasing and there is no resonance at these values.*

Note, furthermore, that as ka *gets large (on the order* $\gamma a/2$*) there are no more solutions to the resonance condition* $\sin(2ka) = -2k/\gamma$*. As a result, the spherical shell potential supports a finite number of quasi-bound states (about* $\gamma a/2\pi$*). This may also be seen in the first figure, where the peaks in negative* δ_0 *are getting smaller, and hence will fall below* $\pi/2$ *(in absolute value) for sufficiently large* ka*.*

2. To a first approximation, the potential that a charged particle feels from a hydrogen atom can be thought of as that due to a positive point charge at the origin (the proton) plus a uniform region of negative charge occupying a sphere of radius a_0 (the electron cloud).

   a) Calculate, in the Born approximation, the differential cross section for the scattering of a charged particle from the hydrogen atom as modeled above (neglect recoil of the hydrogen atom).

   We first need to determine the potential for the electron cloud. Assuming a uniform sphere of charge, the electric field is easily seen to be

$$\mathcal{E}(r) = \begin{cases} -\dfrac{er}{a_0^3}, & r < a_0 \\ -\dfrac{e}{r^2}, & r \geq a_0 \end{cases}$$

(ie it grows linearly inside the sphere, but falls off as an inverse square law outside). Integrating to find the electric potential, $\varphi(r) = -\int_\infty^r \mathcal{E}(r')\,dr'$*, we obtain*

$$\varphi(r) = \begin{cases} \dfrac{e}{a_0}\left[\dfrac{1}{2}\left(\dfrac{r}{a_0}\right)^2 - \dfrac{3}{2}\right], & r < a_0 \\ -\dfrac{e}{r}, & r \geq a_0 \end{cases}$$

To this, we have to add the electric potential from the proton, e/r*. We see that the total potential vanishes outside of* a_0 *(since the hydrogen atom is neutral after all). For interaction with a particle of charge* $Q = Ne$*, the potential (charge times electric potential) is then*

$$V(r) = \frac{Ne^2}{a_0}\left[\frac{1}{2}\left(\frac{r}{a_0}\right)^2 - \frac{3}{2} + \frac{a_0}{r}\right], \qquad r < a_0 \text{ only}$$

All we need to do now is to substitute this into the expression for the Born amplitude

$$\begin{aligned} f_{\text{Born}}(\theta) &= -\frac{2m}{\hbar^2}\int_0^\infty V(r)\frac{\sin qr}{qr}\,r^2\,dr \\ &= -\frac{2mNe^2}{\hbar^2 a_0}\int_0^{a_0}\left[\frac{1}{2}\left(\frac{r}{a_0}\right)^2 - \frac{3}{2} + \frac{a_0}{r}\right]\frac{\sin qr}{qr}\,r^2\,dr \end{aligned}$$

where $q = 2k\sin\frac{\theta}{2}$*. We perform a change of variables,* $x = qr$*, to find*

$$f_{\text{Born}}(\theta) = -\frac{2mNe^2}{\hbar^2 q^2}\int_0^{qa_0}\left[\frac{1}{2(qa_0)^3}x^3 - \frac{3}{2(qa_0)}x + 1\right]\sin x\,dx$$

Using

$$\begin{aligned} \int \sin x\,dx &= -\cos x \\ \int x\sin x\,dx &= -x\cos x + \sin x \\ \int x^3\sin x\,dx &= -x(x^2-6)\cos x + 3(x^2-2)\sin x \end{aligned}$$

we find

$$f_{\text{Born}}(\theta) = -\frac{2mNe^2}{\hbar^2 q^4 a_0^2}\left[(qa_0)^2 + 3\cos(qa_0) - 3\frac{\sin(qa_0)}{qa_0}\right]$$

The differential cross section is thus

$$\frac{d\sigma}{d\Omega} = |f_{\text{Born}}(\theta)|^2 = \frac{4m^2N^2e^4}{\hbar^4 q^8 a_0^4}\left[(qa_0)^2 + 3\cos(qa_0) - 3\frac{\sin(qa_0)}{qa_0}\right]^2$$

b) What is the form of the differential cross section for low energy? Compare with the pure Coulomb cross section.

For low energies, $qa_0 \ll 1$, we may expand

$$\left[(qa_0)^2 + 3\cos(qa_0) - 3\frac{\sin(qa_0)}{qa_0}\right] = \tfrac{1}{10}(qa_0)^4 + \cdots$$

(note that the first two orders cancel in the Taylor expansion), to obtain

$$\frac{d\sigma}{d\Omega} \approx \frac{m^2N^2e^4a_0^4}{25\hbar^4} \qquad (qa_0 \ll 1)$$

This may be written more suggestively by using the expression for the Bohr radius, $a_0 = \hbar^2/me^2$ to obtain

$$\frac{d\sigma}{d\Omega} \approx \frac{N^2}{25}a_0^2$$

which has the form of a hard-sphere cross section (at low energies), with scattering length $a = Na_0/5$. This is certainly not the pure Coulomb cross section. But there is no reason to expect it to be either. At low energies, we do not expect to probe inside the electron cloud, and the hydrogen atom looks essentially like a hard sphere. We can think of the Coulomb potential as being perfectly screened by the hydrogen electron at large distance scales (corresponding to low energies).

c) Show that the differential cross section becomes more and more like a pure Coulomb cross section as the energy of the incident particle increases. Explain why this happens.

When the energy is raised, we need to consider the limit $qa_0 \gg 1$. In this case, the expansion

$$\left[(qa_0)^2 + 3\cos(qa_0) - 3\frac{\sin(qa_0)}{qa_0}\right] = (qa_0)^2 + \cdots$$

is straightforward. This gives

$$\frac{d\sigma}{d\Omega} \approx \frac{4m^2N^2e^4}{\hbar^4q^4} = \frac{m^2N^2e^4}{4(\hbar k)^4\sin^4(\theta/2)} \qquad (qa_0 \gg 1)$$

which is just the Rutherford formula for Coulomb scattering. At high energies, we can probe at smaller distances (given by the Compton wavelength). So in this case we can penetrate the electron cloud and see the bare (unscreened) Coulomb potential of the proton. This is why the limiting case looks more and more like Coulomb scattering.

3. We wish to find an approximate expression for the s-wave phase shift, δ_0, for scattering of low energy particles from the potential

$$V(r) = \frac{C}{r^4}, \qquad C > 0$$

a) For low energies, $k \approx 0$, the radial Schrödinger equation for $\ell = 0$ may be approximated by dropping the energy:

$$\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{2mC}{\hbar^2 r^4}\right] R^<_{\ell=0}(r) = 0$$

By making the transformation

$$R(r) = \frac{1}{\sqrt{r}}\phi(r), \qquad r = \frac{i}{\hbar}\frac{\sqrt{2mC}}{x}$$

show that the the radial equation may be solved in terms of Bessel functions. Find the appropriate solution, taking into account behavior at $r = 0$.

This part is fairly straightforward, and just involves a few transformations. For $R(r) = r^{-1/2}\phi(r)$, *we find*

$$\frac{dR}{dr} = \frac{1}{\sqrt{r}}\left(\frac{d\phi}{dr} - \frac{1}{2r}\phi\right), \qquad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = r^{3/2}\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{4r^2}\phi\right)$$

Hence the radial equation becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \left(\frac{1}{4r^2} + \frac{2mC}{\hbar^2 r^4}\right)\phi = 0 \tag{6}$$

Now transforming from r *to* x, *we find*

$$\frac{d}{dr} = -\frac{i}{\hbar}\frac{\sqrt{2mC}}{r^2}\frac{d}{dx} = \frac{i\hbar}{\sqrt{2mC}}x^2\frac{d}{dx}$$

so that (6) *becomes*

$$\left[x^2\frac{d}{dx}x^2\frac{d}{dx} - x^3\frac{d}{dx} + (x^4 - \tfrac{1}{4}x^2)\right]\phi = 0$$

or

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + (x^2 - \tfrac{1}{4})\phi = 0$$

This is simply Bessel's equation of half-integer order $\nu = 1/2$. *As a result, the general solution may be written in terms of spherical Bessel functions*

$$\phi(x) = \sqrt{x}[Aj_0(x) + Bn_0(x)]$$

or spherical Hankel functions

$$\phi(x) = \sqrt{x}[Ah_0^{(1)}(x) + Bh_0^{(2)}(x)]$$

To understand the behavior of ϕ as $r \to 0$, note that this limit corresponds to $x \to i\infty$. Since $h_0^{(1)}(x) = -ie^{ix}/x$, it is converted into a decaying exponential (and is hence safe). $h_0^{(2)}(x)$, on the other hand, blows up, and should be rejected. So, up to a few constants (which may be absorbed in A), we find

$$R(r) = \sqrt{x}\phi(x) = Axh_0^{(1)}(x) = -iAe^{ix}, \qquad x = \frac{i}{\hbar}\frac{\sqrt{2mC}}{r}$$

Since the overall normalization is irrelevant, we take simply

$$R_{\ell=0}^{<}(r) = e^{-\sqrt{2mC}/\hbar r} \tag{7}$$

So in the end this is a very simple result, and no longer even resembles a Bessel function!

b) By matching this to $R_{\ell=0}^{>}(r)$ at $r = a$ (where a is chosen such that $\hbar a \gg \sqrt{2mC}$ and $ka \ll 1$), show that

$$\delta_0 = -\frac{k\sqrt{2mC}}{\hbar}$$

(which is independent of a).

We may match at $r = a$ by computing the logarithmic derivative of (7)

$$\beta_0 = \frac{aR^{<\prime}(a)}{R^{<}(a)} = a\frac{d}{dr}\log R^{<}(r)\bigg|_{r=a} = \frac{\sqrt{2mC}}{\hbar a}$$

The phase shift is then given by

$$\cot\delta_0 = \frac{ka\, n_0'(ka) - \beta_0\, n_0(ka)}{ka\, j_0'(ka) - \beta_0\, j_0(ka)}$$

We have seen this expression before. Expanding for $ka \ll 1$, we find

$$\cot\delta_0 \approx -\frac{1+\beta_0}{ka\beta_0} \approx -\frac{1}{ka\beta_0} = -\frac{\hbar}{k\sqrt{2mC}}$$

where we have also used the fact that $\hbar a \gg \sqrt{2mC}$. Since the expression for $\cot\delta_0$ *is very large, this indicates that*

$$\delta_0 \approx -\frac{k\sqrt{2mC}}{\hbar}$$

which is the desired result. This problem demonstrates that, while we had to pick a location $r = a$ to match the solutions, in the end the result is independent of a, which is physically reasonable. This provides at least a simple check on the validity of the answer. Of course, if we had a physical barrier at $r = a$ (like the hard sphere or spherical square well), then the result should depend on a. But here a never shows up in the potential $V(r)$, and hence should not show up in the answer.