

Homework Set #9 – Due Friday, March 21

1. We consider first order time dependent perturbation theory for a Hamiltonian  $H(t) = H_0 + V(t)$ . Suppose that a system is initially in an eigenstate  $|s\rangle$  of  $H_0$  at time  $t_0$ . Let  $\mathcal{O}$  be some observable of the system. Show that the expectation value of  $\mathcal{O}$  at time  $t$  is given to first order in  $V(t)$  by

$$\langle \psi(t) | \mathcal{O} | \psi(t) \rangle = \langle s | \mathcal{O} | s \rangle - \frac{i}{\hbar} \int_{t_0}^t \langle s | [\tilde{\mathcal{O}}(t), \tilde{V}(t')] | s \rangle dt'$$

where  $\tilde{\mathcal{O}}(t)$  and  $\tilde{V}(t)$  are in the interaction picture.

2. Apply first-order time dependent perturbation theory to a forced harmonic oscillator

$$H = (a^\dagger a + \frac{1}{2})\hbar\omega + f(t)a + f^*(t)a^\dagger$$

which is initially in the ground state, and compare the transition probability with the exact result

$$P_{n \leftarrow 0} = \frac{1}{n!} \left| \frac{g(\omega)}{\hbar} \right|^{2n} e^{-|g(\omega)/\hbar|^2}$$

where

$$g(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t'} f(t') dt'$$

is the Fourier transform of the generalized force  $f(t)$ . Calculate the energy transfer to the oscillator exactly and also in perturbation theory. Show that the energies agree, even though the probabilities do not (see Exercise 19.2).

3. This is similar to Sakurai, Chapter 5, Problem 25. The unperturbed Hamiltonian of a two-state system is represented by

$$H_0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix}$$

There is, in addition, a time-dependent perturbation

$$V(t) = \lambda \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix} \quad (\lambda \text{ real})$$

- a) At  $t = 0$  the system is known to be in the first state, represented by  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Using time-dependent perturbation theory and assuming that  $E_1^0 - E_2^0$  is not close to  $\pm\hbar\omega$ , derive an expression for the probability for the system to be found in the second state represented by  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  as a function of  $t$  (for  $t > 0$ ).
- b) Why is this procedure not valid when  $E_1^0 - E_2^0$  is close to  $\pm\hbar\omega$ ?

- c) This system may be solved exactly, since it corresponds to a magnetic resonance Hamiltonian. Find the exact transition probability, and compare with the answer to part a).
4. This is based on Sakurai, Chapter 5, Problem 28. A hydrogen atom is initially in its ground state ( $1s$ ). At time  $t = 0$  we turn on a spatially uniform electric field as follows:

$$\vec{E}(t) = \begin{cases} 0 & t < 0 \\ \mathcal{E}_0 e^{-t/\tau} \hat{z} & t \geq 0 \end{cases}$$

- a) Using first-order time dependent perturbation theory, compute the probability for the atom to be found in each of the three  $2p$  states at time  $t \gg \tau$ . You need not evaluate the radial integrals, but perform all other integrations.
- b) What would happen if instead the atom was in the  $2s$  state to begin with?