Physics 512 **Winter 2003**

Homework Set #11 – Due Friday, April 4

1. This is similar to Sakurai, Chapter 6, Problem 7. Two identical spin-$\frac{1}{2}$ fermions are placed in a one-dimensional infinite square well of size L

$$V(x) = \begin{cases} \infty, & x < 0 \text{ or } x > L; \\ 0, & 0 < x < L \end{cases}$$

We assume that the two particles interact mutually via a δ-function potential given by

$$V_{\rm int}(x_1, x_2) = -\lambda\delta(x_1 - x_2)$$

Find the (complete, not just ground state) eigenenergies of this system, including the first order energy shift from $V_{\rm int}$. Take into account both triplet and singlet spin combinations.

The single particle states have the familiar form

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \qquad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

where $n = 1, 2, 3, \ldots$. *Hence for identical fermions, we may form the singlet and triplet combinations*

$$\psi_s = \frac{1}{\sqrt{2}}\big[\phi_{n_1}(x_1)\phi_{n_2}(x_2) + \phi_{n_2}(x_1)\phi_{n_1}(x_2)\big]\chi_{\rm singlet}$$
$$\psi_t = \frac{1}{\sqrt{2}}\big[\phi_{n_1}(x_1)\phi_{n_2}(x_2) - \phi_{n_2}(x_1)\phi_{n_1}(x_2)\big]\chi_{\rm triplet}$$

all with unperturbed energy

$$E^{(0)}_{n_1,n_2} = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2)$$

Note that only the singlet combination survives when $n_1 = n_2$. *Furthermore, in this case, the normalization factor* $1/\sqrt{2}$ *must be replaced by* $1/2$ *(since* ϕ_{n_1} *and* ϕ_{n_2} *are no longer orthogonal when* $n_1 = n_2$*).*

Applying first order time independent perturbation theory with $V_{\rm int}$, *we find*

$$\begin{aligned} E^{(1)}_{n_1,n_2} &= \langle\psi|V_{\rm int}|\psi\rangle \\ &= -\frac{\lambda}{2}\left(\frac{2}{L}\right)^2\int_0^L\!\!\int_0^L\left[\sin\frac{n_1\pi x_1}{L}\sin\frac{n_2\pi x_2}{L} \pm \sin\frac{n_2\pi x_1}{L}\sin\frac{n_1\pi x_2}{L}\right]^2 \\ &\qquad\qquad \times\,\delta(x_1 - x_2)\,dx_1dx_2 \\ &= -\frac{2\lambda}{L^2}\int_0^L\left[\sin\frac{n_1\pi x_1}{L}\sin\frac{n_2\pi x_1}{L} \pm \sin\frac{n_2\pi x_1}{L}\sin\frac{n_1\pi x_1}{L}\right]^2\,dx_1 \end{aligned}$$

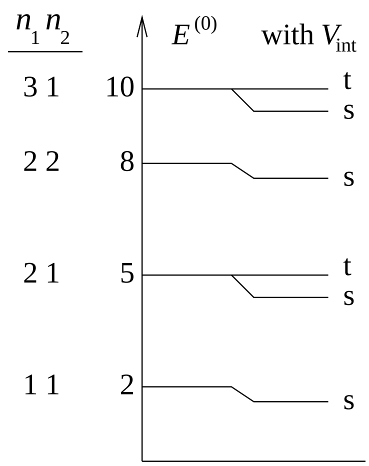
This clearly vanishes for the space antisymmetric combination. Hence only the singlet combination picks up an energy shift at first order. We find

$$
\begin{aligned}
E^{(1)}_{n_1,n_2} &= -\frac{8\lambda}{L^2}\int_0^L \sin^2\frac{n_1\pi x_1}{L}\sin^2\frac{n_2\pi x_1}{L}\,dx_1\\
&= -\frac{2\lambda}{L^2}\int_0^L\left(1-\cos\frac{2n_1\pi x_1}{L}\right)\left(1-\cos\frac{2n_2\pi x_1}{L}\right)dx_1\\
&= -\frac{2\lambda}{L^2}\int_0^L\left[1-\cos\frac{2n_1\pi x_1}{L}-\cos\frac{2n_2\pi x_1}{L}+\frac{1}{2}\cos\frac{2(n_1+n_2)\pi x_1}{L}\right.\\
&\qquad\qquad\qquad\qquad\qquad\left.+\frac{1}{2}\cos 2(n_1-n_2)\pi x_1 L\right]dx_1\\
&= -\frac{2\lambda}{L}(1+\tfrac{1}{2}\delta_{n_1,n_2})
\end{aligned}
$$

(since n_1 and n_2 are positive integers and we are integrating over complete periods of cos*). Reintroducing a factor of 1/2 for the case when $n_1 = n_2$ (to correct for the normalization), we find the first order energy shifts*

$$
E^{(1)}_{n_1,n_2} = \begin{cases} -2\lambda/L, & n_1 \neq n_2\\ -3\lambda/2L, & n_1 = n_2\end{cases} \qquad \textit{(singlet only)}
$$

Thus the low-lying energy spectrum looks like



(where t and s denote triplet and singlet states). It is straightforward to understand why only the singlet states feel the perturbation; for the triplet, the space antisymmetric wave-function vanishes when $x_1 = x_2$, and as a result is unaffected by V_{int}.

2. In the partial wave expansion, the scattering cross section σ_{tot} may be expressed in terms of the phase shifts δ_ℓ.

   a) Show that, for the scattering of two identical spinless bosons, the cross section is given by

$$
\sigma_{\text{tot}} = \frac{16\pi}{k^2}\sum_{\ell\ \text{even}}(2\ell+1)\sin^2\delta_\ell
$$

Recall that the scattering wavefunction for two identical particles may be written as the symmetric or antisymmetric (as appropriate) combination

$$\psi_k(\vec{r}) = e^{ikz} \pm e^{-ikz} + (f_k(\theta) \pm f_k(\pi - \theta))\frac{e^{ikr}}{r}$$

where a partial wave decomposition gives

$$f_k(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$$

The (anti-)symmetrized amplitude is thus

$$\begin{aligned} f_k(\theta) \pm f_k(\pi-\theta) &= \frac{1}{k}\sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell (P_\ell(\cos\theta) \pm P_\ell(-\cos\theta)) \\ &= \frac{2}{k}\sum_{\substack{\ell \text{ even} \\ \text{or} \\ \ell \text{ odd}}} (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) \end{aligned}$$

(since the Legendre polynomials $P_\ell(\zeta)$ are even or odd functions of ζ depending on whether ℓ is even or odd). For identical spin-0 bosons, we take the symmetric combination (ℓ even). This results in the total cross section

$$\sigma_{\text{tot}} = \frac{16\pi}{k^2}\sum_{\ell \text{ even}} (2\ell+1)\sin^2\delta_\ell$$

Note that this differs from the non-identical particles case in that the pre-factor is four times as large while the sum is only over every other value of ℓ. Roughly speaking, this indicates the cross section would end up being about twice as large. This factor of two corresponds simply to the detection of either one of the two particles without regard to where it came from (whereas the non-identical cross section corresponds only to the detection of, say, the first particle).

The fact that the sum is only over even values of ℓ is the same as the statement that identical spin-0 bosons can only be in states of even orbital angular momentum.

b) What is the expression for the cross section for two identical spin-$\frac{1}{2}$ fermions?

We assume the scattering potential is spin independent. In this case, for identical spin-$\frac{1}{2}$ fermions, we may consider either singlet or triplet scattering. Since the total (spin plus orbital) wavefunction must be antisymmetric, we find

$$\begin{aligned} \sigma_{\text{tot}}^{\text{singlet}} &= \frac{16\pi}{k^2}\sum_{\ell \text{ even}} (2\ell+1)\sin^2\delta_\ell \\ \sigma_{\text{tot}}^{\text{triplet}} &= \frac{16\pi}{k^2}\sum_{\ell \text{ odd}} (2\ell+1)\sin^2\delta_\ell \end{aligned}$$

For unpolarized particles (25% singlet, 75% triplet), we find

$$\sigma_{\text{tot}}^{\text{unpolarized}} = \frac{4\pi}{k^2}\left(\sum_{\ell\ \text{even}}(2\ell+1)\sin^2\delta_\ell + 3\sum_{\ell\ \text{odd}}(2\ell+1)\sin^2\delta_\ell\right)$$

3. This is based on Merzbacher, Exercise 23.9.

   a) Work out the equal-time commutation relations between the components of $\vec{A}(\vec{r},t)$ and $\vec{E}(\vec{r}\,',t)$.

   We use the expansions

$$\begin{aligned}\vec{A}^{(+)} &= \sqrt{2\pi\hbar c^2}\frac{1}{L^{3/2}}\sum_{\vec{k},\alpha}\frac{1}{\sqrt{\omega_k}}\,a_\alpha(\vec{k})\hat{\epsilon}_k^{(\alpha)}e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\\ \vec{E}^{(+)} &= i\sqrt{2\pi\hbar c^2}\frac{1}{L^{3/2}}\sum_{\vec{k},\alpha}\sqrt{\omega_k}\,a_\alpha(\vec{k})\hat{\epsilon}_k^{(\alpha)}e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\end{aligned}\tag{1}$$

   and similar ones for $\vec{A}^{(-)}$ and $\vec{E}^{(-)}$. Since the only operators that do not commute are

$$[a_\alpha(\vec{k}), a^\dagger_{\alpha'}(\vec{k}')] = \delta_{\alpha,\alpha'}\delta_{\vec{k},\vec{k}'}$$

   we must combine positive frequency modes with negative frequency ones. Thus, for equal times but different positions

$$\begin{aligned}[\vec{A}(\vec{r},t),\vec{E}(\vec{r}\,',t)] &= [\vec{A}^{(+)}(\vec{r},t),\vec{E}^{(-)}(\vec{r}\,',t)] + [\vec{A}^{(-)}(\vec{r},t),\vec{E}^{(+)}(\vec{r}\,',t)]\\ &= -2\pi i\hbar c\frac{1}{L^3}\sum_{\substack{\vec{k},\alpha\\ \vec{k}',\alpha'}}\sqrt{\frac{\omega_{k'}}{\omega_k}}[a_\alpha(\vec{k}),a^\dagger_{\alpha'}(\vec{k}')]\hat{\epsilon}_k^{(\alpha)}\hat{\epsilon}_{k'}^{(\alpha')}e^{i(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r}\,'-(\omega_k-\omega_{k'})t)}\\ &\quad + 2\pi i\hbar c\frac{1}{L^3}\sum_{\substack{\vec{k},\alpha\\ \vec{k}',\alpha'}}\sqrt{\frac{\omega_{k'}}{\omega_k}}[a^\dagger_\alpha(\vec{k}),a_{\alpha'}(\vec{k}')]\hat{\epsilon}_k^{(\alpha)}\hat{\epsilon}_{k'}^{(\alpha')}e^{-i(\vec{k}\cdot\vec{r}-\vec{k}'\cdot\vec{r}\,'-(\omega_k-\omega_{k'})t)}\\ &= -2\pi i\hbar c\frac{1}{L^3}\sum_{\vec{k},\alpha}\hat{\epsilon}_k^{(\alpha)}\hat{\epsilon}_k^{(\alpha)}\left(e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} + e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\right)\end{aligned}$$

   Since we are summing over all $\vec{k}$, we may let $\vec{k}\to-\vec{k}$ to handle the second term. Note that the appropriate polarization vectors would be $\hat{\epsilon}_{-k}^{(\alpha)}$. However we may use identical polarization basis for both $\vec{k}$ and $-\vec{k}$. With this simplification, we find

$$[\vec{A}(\vec{r},t),\vec{E}(\vec{r}\,',t)] = -4\pi i\hbar c\frac{1}{L^3}\sum_{\vec{k},\alpha}\hat{\epsilon}_k^{(\alpha)}\hat{\epsilon}_k^{(\alpha)}e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

For a fixed $\vec{k}$, we sum over two orthonormal polarization vectors (both transverse to k). In particular, $\hat{k}$, $\hat{\epsilon}_k^{(1)}$ and $\hat{\epsilon}_k^{(2)}$ form an orthonormal basis. Thus (in dyadic notation)

$$\hat{\epsilon}_k^{(1)}\hat{\epsilon}_k^{(1)} + \hat{\epsilon}_k^{(2)}\hat{\epsilon}_k^{(2)} + \hat{k}\hat{k} = I \quad (\textit{the } 3\times 3 \textit{ identity matrix})$$

This allows us to rewrite the commutator of $\vec{A}$ and $\vec{E}$ as

$$[\vec{A}(\vec{r},t), \vec{E}(\vec{r}\,',t)] = -4\pi i\hbar c\frac{1}{L^3}\sum_{\vec{k}}(I - \hat{k}\hat{k})e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

which is independent of the details of the polarization vectors. In component form, this reads

$$[A_i(\vec{r},t), E_j(\vec{r}\,',t)] = -4\pi i\hbar c\frac{1}{L^3}\sum_{\vec{k}}(\delta_{ij} - \hat{k}_i\hat{k}_j)e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad (2)$$

Note that if the $\hat{k}_i\hat{k}_j$ term were absent, the sum could be rewritten as

$$[A_i(\vec{r},t), E_j(\vec{r}\,',t)] = -4\pi i\hbar c\,\delta_{ij}\,\delta^3(\vec{r}-\vec{r}\,')$$

For this reason, it is convenient to define δ^{tr}, a 'transverse' δ-function, in such a way that the commutator expression (2) is given by

$$[A_i(\vec{r},t), E_j(\vec{r}\,',t)] = -4\pi i\hbar c\,\delta_{ij}^{\mathrm{tr}}(\vec{r}-\vec{r}\,') \qquad (3)$$

This is of course simply a definition of δ^{tr}. However it does capture the physical idea that $I - \hat{k}\hat{k}$ projects onto the polarization states transverse to the direction of propagation of the photon. So (3) essentially indicates that the photon has only two transverse degrees of freedom instead of three.

In classical Lagrangian field theory, we may consider the vector potential $\vec{A}$ as a set of generalized coordinates with generalized momenta

$$\vec{\Pi} = \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\vec{A} = -\frac{1}{4\pi c}\vec{E}$$

As a result, (3) is simply the statement

$$[A_i(\vec{r},t), \Pi_j(\vec{r}\,',t)] = i\hbar\delta_{ij}^{\mathrm{tr}}(\vec{r}-\vec{r}\,')$$

which is a generalization of the basic commutation relation $[x,p] = i\hbar$. This is the reason why we claim that the mode expansion of the vector potential, (1), is the appropriate one for quantum field theory.

b) For $\vec{B} = \vec{\nabla}\times\vec{A}$, what does this say about simultaneous measurements of electric and magnetic fields? Note that you may expect to find the derivative of a delta-function to turn up.

In component form, the magnetic field is given by $B_i = \epsilon_{ijk}\partial_{r^j} A_k(\vec{r})$. Hence, using (2), we find

$$\begin{aligned}[B_i(\vec{r},t), E_j(\vec{r}\,',t)] &= \epsilon_{ikl}\partial_{r^k}[A_l(\vec{r},t), E_j(\vec{r}\,',t)] \\ &= -4\pi i\hbar c\,\epsilon_{ikl}\partial_{r^k}\frac{1}{L^3}\sum_{\vec{k}}(\delta_{lj} - \hat{k}_l\hat{k}_j)e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\end{aligned}$$

Since ∂_{r^k} brings down a factor of ik_k from the exponential, and because of the ϵ_{ikl} tensor, the second term in the parentheses vanish. This gives

$$\begin{aligned}[B_i(\vec{r},t), E_j(\vec{r}\,',t)] &= 4\pi i\hbar c\,\epsilon_{ijk}\partial_{r^k}\frac{1}{L^3}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \\ &= 4\pi i\hbar c\,\epsilon_{ijk}\partial_{r^k}\delta^3(\vec{r}-\vec{r}\,')\end{aligned}$$

As a result, $\vec{E}$ and $\vec{B}$ do not commute at the same point in space, and hence they cannot be simultaneous observables. [Actually, only perpendicular components do not commute, eg E_x and B_y].

4. Sakurai, Chapter 5, Problem 40. A hydrogen atom in the $2p$ state will decay to the $1s$ ground state through spontaneous emission. Compute the rate of spontaneous emission in the electric dipole approximation and show that this predicts a lifetime for the $2p$ state of 1.6×10^{-9} s. [Ignore electron spin].

   For spontaneous emission with only a single photon in the final state ($n_f = 1$), we may use the decay rate expression

$$w = \frac{\alpha}{2\pi m^2c^2}\,\omega|\langle f|e^{-i\vec{k}\cdot\vec{r}}\hat{\epsilon}\cdot\vec{p}|i\rangle|^2\,d\Omega \approx \frac{\alpha}{2\pi m^2c^2}\,\omega|\langle f|\hat{\epsilon}\cdot\vec{p}|i\rangle|^2\,d\Omega$$

   (in the electric dipole approximation). Because of rotational invariance, we may choose a basis such that the initial state is in the $m = 0$ or $|210\rangle$ state. In this case,

$$\langle f|\hat{\epsilon}\cdot\vec{p}|i\rangle = \langle 100|\hat{\epsilon}\cdot\vec{p}|210\rangle = im\omega\langle 100|\hat{\epsilon}\cdot\vec{r}|210\rangle = im\omega\hat{\epsilon}_z\langle 100|z|210\rangle$$

   (because we must have $\Delta m = 0$). Thus

$$w = \frac{\alpha\omega^3}{2\pi c^2}\,(\hat{\epsilon}_z)^2|\langle 100|z|210\rangle|^2\,d\Omega$$

   and we are left with calculating this matrix element. Using

$$\psi_{100} = \sqrt{\frac{1}{\pi a_0^3}}e^{-r/a_0}, \qquad \psi_{210} = \sqrt{\frac{1}{2\pi a_0^3}}\frac{r}{4a_0}e^{-r/2a_0}\cos\theta$$

we find

$$\begin{aligned}
\langle 100|z|210\rangle &= \frac{1}{\sqrt{2}\pi a_0^3}\int e^{-r/a_0}(r\cos\theta)\frac{r}{4a_0}e^{-r/2a_0}\cos\theta\, r^2 dr d\cos\theta d\phi \\
&= \frac{1}{4\sqrt{2}\pi a_0^4}\int_0^\infty r^4 e^{-3r/2a_0}\,dr \times 2\pi\int_{-1}^1 \cos^2\theta\, d\cos\theta \\
&= \frac{1}{3\sqrt{2}a_0^4}\left(\frac{2a_0}{3}\right)^5\int_0^\infty u^4 e^{-u}\,du \\
&= \frac{a_0}{3\sqrt{2}}\left(\frac{2}{3}\right)^5 4! = \frac{128\sqrt{2}}{243}a_0
\end{aligned}$$

As a result

$$w = \frac{\alpha a_0^2\omega^3}{2\pi c^2}\frac{2^{15}}{3^{10}}(\hat{\epsilon}_z)^2\, d\Omega$$

To find the total decay rate, we must sum over polarization states and integrate over angles of the decay photon. For a photon emitted with angle θ from the $\hat{z}$-axis, we may pick find an orthonormal set of linear polarizations where $\epsilon_{1z} = 0$ and $\epsilon_{2z} = \sin\theta$. Thus

$$\int(\hat{\epsilon}_z)^2\, d\Omega = \int\sin^2\theta\, d\Omega = \frac{8\pi}{3}$$

so that

$$w = \frac{\alpha a_0^2\omega^3}{c^2}\frac{2^{17}}{3^{11}}$$

Since $\hbar\omega = E_{2s} - E_{1s} = \frac{3}{4}E_0$, and $E_0 = e^2/2a_0$, we find

$$w = \frac{\alpha a_0^2 E_0^3}{\hbar^3 c^2}\frac{2^{11}}{3^8} = \frac{\alpha^3 E_0}{\hbar}\frac{2^9}{3^8} = 0.078\frac{\alpha^3 E_0}{\hbar}$$

Putting in numbers, we find

$$w = 0.078 \times \frac{1}{137.036^3} \times \frac{13.6\,\text{eV}}{6.58\times 10^{-16}\,\text{eV s}} = (1.6\times 10^{-9}\,\text{s})^{-1}$$