Physics 512 **Winter 2003**

Homework Set #12 – Solutions

1. The differential cross section for the ejection of an electron with momentum $\vec{\mathbf{k}}_f$ by an incident photon of momentum $\vec{k}$ ($\omega = c|\vec{k}\,|$) and polarization $\hat{\epsilon}$ (the photoelectric effect) may be written as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha|\vec{\mathbf{k}}_f|}{2\pi m\hbar\omega} L^3 |\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle|^2$$

where the matrix element refers to the initial (bound) and final (free plane wave) electron states. Derive this expression using the quantum theory of radiation (instead of the classical treatment shown in class).

We begin by computing the transition rate for this process to occur. For the rate, Fermi's golden rule for harmonic perturbations gives

$$w = \frac{2\pi}{\hbar}|\langle f;0|V|i;\{n_{\vec{k}}^{(\alpha)} = 1\}\rangle|^2 \rho_f(E_f)$$

Here we have used a notation where i and f denote the initial and final electron state, while the remaining pieces are a shorthand label for the photons in the Fock space. For absorption of a photon, we have

$$V = \frac{e}{mc}\vec{A}^{(+)}\cdot\vec{p} = \frac{e}{mc}\sqrt{2\pi\hbar c^2}\frac{1}{L^{3/2}}\sum_{\vec{k},\alpha}\frac{1}{\sqrt{\omega}}a_\alpha(\vec{k}\,)\vec{p}\cdot\hat{\epsilon}^{(\alpha)}e^{i\vec{k}\cdot\vec{r}}$$

Evaluating the matrix element in the photon Fock space, we see that the lowering operator annihilates the initial photon, leaving the Fock vacuum (which is normalized so that $\langle 0|0\rangle = 1$). This selects out only one particular term in the sum (corresponding to the single initial state photon). As a result

$$\langle f;0|V|i;\{n_{\vec{k}}^{(\alpha)} = 1\}\rangle = \frac{e}{mc}\sqrt{\frac{2\pi\hbar c^2}{\omega L^3}}\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle$$

Squaring this and using Fermi's golden rule yields

$$\begin{aligned} w &= \frac{(2\pi)^2e^2}{m^2\omega L^3}|\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle|^2 \\ &= \frac{(2\pi)^2e^2}{m^2\omega L^3}\frac{m|\vec{\mathbf{k}}_f|}{\hbar^2}\left(\frac{L}{2\pi}\right)^3|\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle|^2\rho_f(E_f) \\ &= \frac{\alpha c|\vec{\mathbf{k}}_f|}{2\pi m\hbar\omega}|\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle|^2 d\Omega \end{aligned}$$

where, for the final state, we have used the free electron density of states

$$\rho(E_f) = \frac{m|\vec{\mathbf{k}}_f|}{\hbar^2}\left(\frac{L}{2\pi}\right)^3 d\Omega$$

Finally, this may be converted to a differential cross section by dividing by the incident (single photon) flux c/L^3. This gives the desired result

$$\frac{d\sigma}{d\Omega} = \frac{\alpha|\vec{\mathbf{k}}_f|}{2\pi m\hbar\omega} L^3 |\langle f|e^{i\vec{k}\cdot\vec{r}}\vec{p}\cdot\hat{\epsilon}|i\rangle|^2$$

Note that this matrix element is only for the electron states. Moreover, the final ejected electron wavefunction must be normalized as $\psi \sim 1/L^{3/2}$. Taking this into account, the final result for the cross section will be independent of the box size L.

2. A quantized neutral scalar field (Klein-Gordon field) may be expanded in terms of creation and annihilation operators as follows

$$\phi(\vec{r},t) = \phi^{(+)}(\vec{r},t) + \phi^{(-)}(\vec{r},t) = \sqrt{\hbar c^2}\frac{1}{L^{3/2}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega}}\left[a(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)} + a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

As usual, $a(\vec{k})$ and $a^\dagger(\vec{k})$ satisfy the commutation relation $[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k},\vec{k}'}$ (all others vanish). In addition, ω satisfies the relativistic energy relation

$$\omega = \sqrt{|c\vec{k}|^2 + (mc^2/\hbar)^2}$$

a) Compute the expectation value (two-point function)

$$\langle 0|\phi(\vec{r},t)\phi(\vec{r}\,',t')|0\rangle \equiv \langle \text{vac}|\phi(\vec{r},t)\phi(\vec{r}\,',t')|\text{vac}\rangle$$

You do not have to evaluate the sum over $\vec{k}$.

This two-point function has an important physical meaning. Consider that, to get a non-vanishing amplitude, we may keep track of the "particle number" and note that the vacuum $|0\rangle$ starts with no particles (annihilated by all the $a(\vec{k})$ operators). Then the state $\phi(\vec{r}\,',t')|0\rangle$ is a sum of single particle states because of the creation operators in $\phi^{(-)}(\vec{r}\,',t')$. Finally, $\phi(\vec{r},t)\phi(\vec{r}\,',t')|0\rangle$ may contain both zero and two-particle states. Of course, only the zero-particle state (which is just the vacuum) survives when taking the matrix element with $\langle 0|$. In other words

$$\langle 0|\phi(\vec{r},t)\phi(\vec{r}\,',t')|0\rangle = \langle 0|\phi^{(+)}(\vec{r},t)\phi^{(-)}(\vec{r}\,',t')|0\rangle$$

Furthermore, the particle destroyed by $\phi^{(+)}(\vec{r},t)$ must be the same as the one created by $\phi^{(-)}(\vec{r}\,',t')$. Thus, inserting the explicit expression for $\phi(\vec{r},t)$ into the above, we find

$$\begin{aligned}\langle 0|\phi(\vec{r},t)\phi(\vec{r}\,',t')|0\rangle &= \frac{\hbar c^2}{L^3}\sum_{\vec{k}}\frac{1}{2\omega}\langle 0|a(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\omega t)}a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{r}\,'-\omega t')}|0\rangle \\ &= \frac{\hbar c^2}{L^3}\sum_{\vec{k}}\frac{1}{2\omega}e^{i(\vec{k}\cdot\Delta\vec{r}-\omega\Delta t)}\end{aligned} \tag{1}$$

where $\Delta\vec{r} = \vec{r} - \vec{r}\,'$ *and* $\Delta t = t - t'$. *Of course we must use the relativistic dispersion relation given above for* $\omega(\vec{k}\,)$. *As a result, evaluating the sum is quite unpleasant.*

This two-point function is very closely related to the propagator (or Green's function) for the Klein-Gordon field. After all, as we saw above, in terms of the creation and annihilation operators, the two-point function essentially represents the creation of a particle at the spacetime point $(\vec{r}\,', t')$ *and the destruction of the same particle at the point* $(\vec{r}, t)$ *[this is also why it's called a correlation function—it measures the correlation of the Klein-Gordon field between two spacetime points]. However, to get a physically useful propagator (advanced, retarded or Feynman propagator), we have to be more careful about time ordering issues.*

Finally, note that this expression may be expressed in a relativistically invariant manner. To see this, we first take the continuum limit

$$\langle 0|\phi(\vec{r}, t)\phi(\vec{r}\,', t')|0\rangle = \hbar c^2 \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega}e^{-ik_\mu x^\mu}$$

where $k^\mu = (\omega/c, \vec{k}\,)$ *and then make use of Dirac* δ*-function properties to write*

$$\int dk^0\,\delta(k_\mu k^\mu - (mc/\hbar)^2)\theta(k^0) = \frac{1}{2k^0} = \frac{c}{2\omega}$$

Here $\theta(k^0) = 1$ *for* $k^0 > 0$ *(and 0 otherwise). This trick basically adds in an integral over all energies,* $\int dk^0$, *then imposes the "mass-shell condition" (which is just the relativistic energy relation) as well as a condition to include only positive energies. Putting this together gives*

$$\langle 0|\phi(\vec{r}, t)\phi(\vec{r}\,', t')|0\rangle = 2\pi\hbar c \int \frac{d^4k}{(2\pi)^4}\delta(k_\mu k^\mu - (mc/\hbar)^2)\theta(k^0)e^{-ik_\mu x^\mu} \qquad (2)$$

which is a Lorentz invariant integral.

b) Show that the equal time commutator vanishes

$$\langle 0|[\phi(\vec{r}, t), \phi(\vec{r}\,', t')]|0\rangle = 0 \quad \text{for} \quad t = t'$$

Using (1), *we compute (at equal times)*

$$\langle 0|[\phi(\vec{r}, t), \phi(\vec{r}\,', t)]|0\rangle = \frac{\hbar c^2}{L^3}\sum_{\vec{k}} \frac{1}{2\omega}\left(e^{i\vec{k}\cdot\Delta\vec{r}} - e^{-i\vec{k}\cdot\Delta\vec{r}}\right)$$

Since we are summing over all $\vec{k}$, *we may make the substitution* $\vec{k} \to -\vec{k}$ *in the second term in the sum. Since* $\omega(-\vec{k}\,) = \omega(\vec{k}\,)$, *the sum vanishes, and hence the equal time commutator vanishes. Actually we could have also verified this by computing the commutator of the field operators* $\phi(\vec{r}, t)$ *and* $\phi(\vec{r}\,', t)$ *directly* *without needing to take any expectation values. In this sense, the equal time commutator vanishes identically as an operator statement.*

The vanishing commutator is a statement that we could simultaneously observe the Klein-Gordon field at two distinct points in space (at the same time). This is also a statement of causality—namely that observations at the points $\vec{r}$ and $\vec{r}\,'$ cannot simultaneously influence each other. Actually we can say much more than this. By performing Lorentz transformations, equal times $t = t'$ in one Lorentz frame is no longer equal in a boosted frame. However the commutator may be written in a Lorentz invariant manner according to (2). So if it starts out zero (for equal times), it must remain zero in a boosted frame. Since $\Delta\vec{r}$ is a spacelike interval, and since Lorentz transformations preserve the interval, this implies that the commutator $[\phi(x^\mu), \phi(x'^\mu)]$ vanishes for all spacelike separated points x^μ and x'^μ. In other words, any two measurements cannot influence each other outside the light-cone, and the quantum field theory of a Klein-Gordon field is consistent with causality.

3. Merzbacher, Exercise 24.11. Show that the 4×4 matrices $\vec{\Sigma}$ where

$$\vec{\Sigma} = (\Sigma^{23}, \Sigma^{31}, \Sigma^{12}), \qquad \Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

satisfy the usual commutation relations for Pauli spin matrices. Show that in the standard representation

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

In component form, we write $\Sigma_1 = \Sigma^{23}$, $\Sigma_2 = \Sigma^{31}$ and $\Sigma_3 = \Sigma^{12}$. Since different Dirac matrices anticommute with each other, we find simple expressions for eg

$$\Sigma_1 = \frac{i}{2}[\gamma^2, \gamma^3] = \frac{i}{2}(\gamma^2\gamma^3 - \gamma^3\gamma^2) = i\gamma^2\gamma^3$$

Similarly,

$$\Sigma_2 = i\gamma^3\gamma^1, \qquad \Sigma_3 = i\gamma^1\gamma^2$$

To check the commutation relations for $\vec{\Sigma}$, we examine eg

$$[\Sigma_1, \Sigma_2] = [i\gamma^2\gamma^3, i\gamma^3\gamma^1] = -(\gamma^2\gamma^3\gamma^3\gamma^1 - \gamma^3\gamma^1\gamma^2\gamma^3)$$

In the last term, we may anticommute the γ^3 matrices to the center of the expression

$$[\Sigma_1, \Sigma_2] = -(\gamma^2\gamma^3\gamma^3\gamma^1 - \gamma^1\gamma^3\gamma^3\gamma^2)$$

Noting that we are working with a "mostly negative" metric (ie $(\gamma^3)^2 = -1$), we find

$$[\Sigma_1, \Sigma_2] = \gamma^2\gamma^1 - \gamma^1\gamma^2 = -2\gamma^1\gamma^2 = 2i\Sigma_3$$

Similarly, we can work out the other commutators $[\Sigma_2, \Sigma_3]$ *and* $[\Sigma_3, \Sigma_1]$. *The result is summarized as*

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k$$

which is identical to the ordinary commutation relation for the Pauli matrices, with the substitution $\vec{\sigma} \to \vec{\Sigma}$.

To work out the form of $\vec{\Sigma}$ *in the standard representation, we recall that*

$$\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

Then

$$\Sigma_i = \frac{i}{2}\epsilon_{ijk}\gamma^j\gamma^k = \frac{i}{2}\epsilon_{ijk} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} = -\frac{i}{2}\epsilon_{ijk} \begin{pmatrix} \sigma_j\sigma_k & 0 \\ 0 & \sigma_j\sigma_k \end{pmatrix}$$

Using $\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l$, *this gives the result*

$$\Sigma_i = \tfrac{1}{2}\epsilon_{ijk}\epsilon_{jkl} \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

Thus the $\vec{\Sigma}$ *matrices generalize the spin-*1/2 *Pauli matrices to the case of the Dirac equation.*